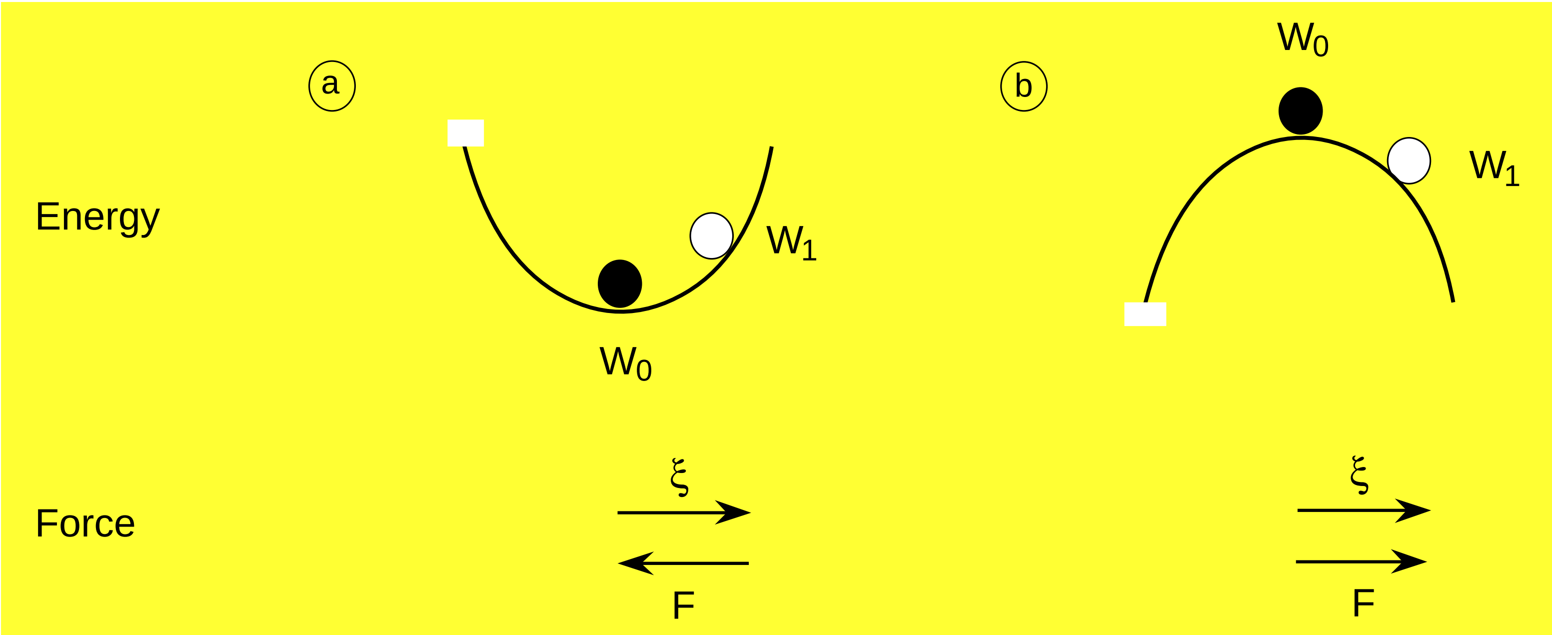
Chapter 6: Spectral Theory

Overview

- **Intuitive approach to stability:** two viewpoints for study of stability, linearization and Lagrangian reduction; [book: Sec. 6.1]
- **Force operator formalism:** equation of motion, Hilbert space, self-adjointness of the force operator; [book: Sec. 6.2]
- **Quadratic forms and variational principles:** expressions for the potential energy, different variational principles, the energy principle; [book: Sec. 6.4]
- **Further spectral issues:** returning to the two viewpoints; [book: Sec. 6.5]
- **Extension to interface plasmas:** boundary conditions, extended variational principles, Rayleigh–Taylor instability. [book: Sec. 6.6]

Two viewpoints

- *How does one know whether a dynamical system is stable or not?*



stable unstable

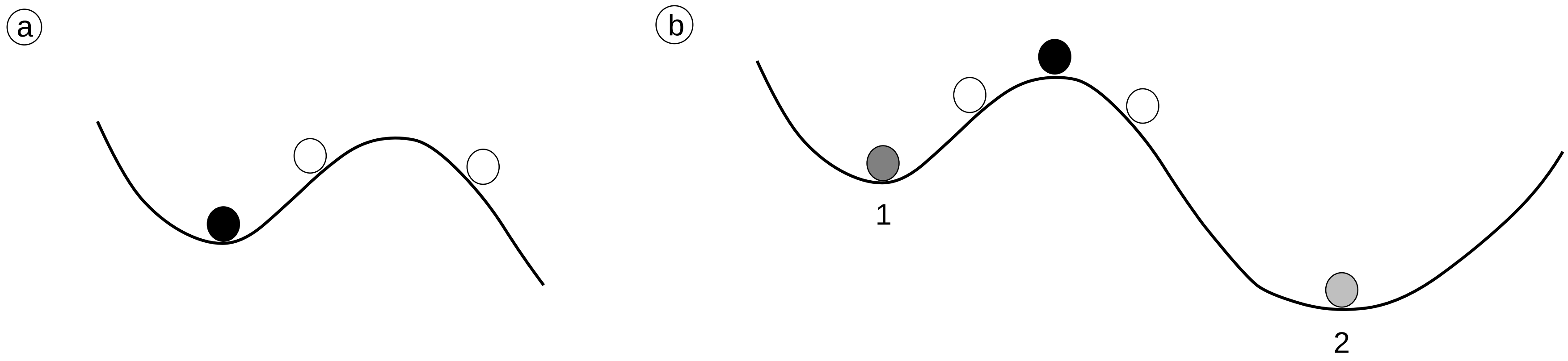
- Method: split the non-linear problem in *static equilibrium* (no flow) and small (linear) *time-dependent perturbations*.
- Two approaches: exploiting variational principles involving *quadratic forms* (energy), or solving *the partial differential equations themselves* (forces).

Aside: nonlinear stability

- Distinct from linear stability, involves *finite amplitude displacements:*

  (a) system can be linearly stable, nonlinearly unstable;

  (b) system can be linearly unstable, nonlinearly stable (e.g. evolving towards the equilibrium states 1 or 2).



- Quite relevant for topic of magnetic confinement, but too complicated at this stage.

Linearization

- Start from *ideal MHD equations:*

$$\rho(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla p + \mathbf{j}\times\mathbf{B} - \rho\nabla\Phi\,, \qquad \mathbf{j} = \nabla\times\mathbf{B}\,, \tag{1}$$

$$\frac{\partial p}{\partial t} = -\mathbf{v}\cdot\nabla p - \gamma p\nabla\cdot\mathbf{v}\,, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B})\,, \qquad \nabla\cdot\mathbf{B} = 0\,, \tag{3}$$

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\mathbf{v})\,. \tag{4}$$

assuming *model I (plasma–wall) BCs:*

$$\mathbf{n}\cdot\mathbf{v} = 0\,, \qquad \mathbf{n}\cdot\mathbf{B} = 0 \qquad \textit{(at the wall)}\,. \tag{5}$$

- Linearize about *static equilibrium* with time-independent ρ_0, p_0, $\mathbf{B}_0$, and $\mathbf{v}_0 = 0$:

$$\mathbf{j}_0\times\mathbf{B}_0 = \nabla p_0 + \rho_0\nabla\Phi\,, \qquad \mathbf{j}_0 = \nabla\times\mathbf{B}_0\,, \qquad \nabla\cdot\mathbf{B}_0 = 0\,, \tag{6}$$

$$\mathbf{n}\cdot\mathbf{B}_0 = 0 \quad \textit{(at the wall)}\,. \tag{7}$$

- Time dependence enters through *linear perturbations* of the equilibrium:

$$\begin{aligned} \mathbf{v}(\mathbf{r},t) &= \mathbf{v}_1(\mathbf{r},t)\,, \\ p(\mathbf{r},t) &= p_0(\mathbf{r}) + p_1(\mathbf{r},t)\,, \\ \mathbf{B}(\mathbf{r},t) &= \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r},t)\,, \\ \rho(\mathbf{r},t) &= \rho_0(\mathbf{r}) + \rho_1(\mathbf{r},t)\,, \end{aligned} \qquad \big(\text{all, except } \mathbf{v}_1\text{: } |f_1(\mathbf{r},t)| \ll |f_0(\mathbf{r})|\,\big)\,. \tag{8}$$

- Inserting in Eqs. (1)–(4) yields *linear equations for $\mathbf{v}_1$, p_1, $\mathbf{B}_1$, ρ_1* (note strange order!):

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \mathbf{j}_1 \times \mathbf{B}_0 + \mathbf{j}_0 \times \mathbf{B}_1 - \rho_1 \nabla \Phi\,, \qquad \mathbf{j}_1 = \nabla \times \mathbf{B}_1\,, \tag{9}$$

$$\frac{\partial p_1}{\partial t} = -\mathbf{v}_1 \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \mathbf{v}_1\,, \tag{10}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0)\,, \qquad \nabla \cdot \mathbf{B}_1 = 0\,, \tag{11}$$

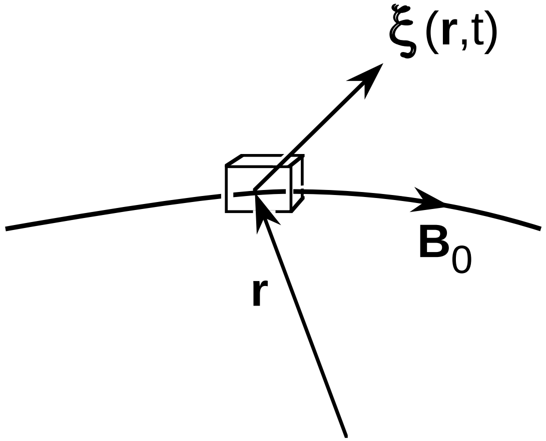
$$\frac{\partial \rho_1}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{v}_1)\,. \tag{12}$$

Since wall fixed, so is $\mathbf{n}$, hence BCs (5) already linear:

$$\mathbf{n} \cdot \mathbf{v}_1 = 0\,, \qquad \mathbf{n} \cdot \mathbf{B}_1 = 0 \qquad \textit{(at the wall).} \tag{13}$$

Lagrangian reduction

- Introduce *Lagrangian displacement vector field* $\boldsymbol{\xi}(\mathbf{r}, t)$*:*
  plasma element is moved over $\boldsymbol{\xi}(\mathbf{r}, t)$ away from the equilibrium position.



$\Rightarrow$ Velocity is time variation of $\boldsymbol{\xi}(\mathbf{r}, t)$ in the comoving frame,

$$\mathbf{v} = \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t} \equiv \frac{\partial \boldsymbol{\xi}}{\partial t} + \mathbf{v} \cdot \nabla \boldsymbol{\xi} \,, \tag{14}$$

involving the Lagrangian time derivative $\dfrac{\mathrm{D}}{\mathrm{D}t}$ (co-moving with the plasma).

- Linear (first order) part relation yields

$$\mathbf{v} \approx \mathbf{v}_1 = \frac{\partial \boldsymbol{\xi}}{\partial t}\,, \tag{15}$$

only involving the Eulerian time derivative (fixed in space).

- Inserting in linearized equations, can directly integrate (12):

$$\frac{\partial \rho_1}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{v}_1) \qquad \Rightarrow \qquad \rho_1 = -\nabla \cdot (\rho_0 \boldsymbol{\xi})\,. \tag{16}$$

Similarly linearized energy (10) and induction equation (11) integrate to

$$p_1 = -\,\boldsymbol{\xi} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \boldsymbol{\xi}\,, \tag{17}$$

$$\mathbf{B}_1 = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) \qquad \big(\text{automatically satisfies } \nabla \cdot \mathbf{B}_1 = 0\,\big)\,. \tag{18}$$

- Inserting these expressions into linearized momentum equation yields

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}\big(p_1(\boldsymbol{\xi}), \mathbf{B}_1(\boldsymbol{\xi}), \rho_1(\boldsymbol{\xi})\big)\,. \tag{19}$$

$\Rightarrow$ **Equation of motion with force operator $\mathbf{F}$.**

Force Operator formalism

- Insert explicit expression for $\mathbf{F}$ $\Rightarrow$ *Newton's law for plasma element:*

$$\mathbf{F}(\boldsymbol{\xi}) \equiv -\nabla\pi - \mathbf{B}\times(\nabla\times\mathbf{Q}) + (\nabla\times\mathbf{B})\times\mathbf{Q} + (\nabla\Phi)\,\nabla\cdot(\rho\boldsymbol{\xi}) = \rho\,\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}\,, \qquad (20)$$

with change of notation (so that we can drop subscripts $_0$ and $_1$):

$$\pi \;\equiv\; p_1 = -\gamma p \nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla p\,, \qquad (21)$$

$$\mathbf{Q} \;\equiv\; \mathbf{B}_1 = \nabla\times(\boldsymbol{\xi}\times\mathbf{B})\,. \qquad (22)$$

- Geometry (plane slab, cylinder, torus, etc.) defined by shape wall, through BC:

$$\mathbf{n}\cdot\boldsymbol{\xi} = 0 \qquad \text{(at the wall)}\,. \qquad (23)$$

- Now count: three 2nd order PDEs for vector $\boldsymbol{\xi}$ $\Rightarrow$ sixth order *Lagrangian* system; originally: eight 1st order PDEs for ρ_1, $\mathbf{v}_1$, p_1, $\mathbf{B}_1$ $\Rightarrow$ eight order *Eulerian* system.

- Third component of $\mathbf{B}_1$ is redundant ($\nabla\cdot\mathbf{B}_1 = 0$), and equation for ρ_1 produces trivial Eulerian entropy mode $\omega_E = 0$ (with $\rho_1 \neq 0$, but $\mathbf{v}_1 = 0$, $p_1 = 0$, $\mathbf{B}_1 = 0$).

  $\Rightarrow$ Neglecting this mode, Lagrangian and Eulerian representation equivalent.

Ideal MHD spectrum

- Consider *normal modes:*

$$\boldsymbol{\xi}(\mathbf{r}, t) = \hat{\boldsymbol{\xi}}(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}\omega t} \,. \tag{24}$$

  ⇒ Equation of motion becomes eigenvalue problem:

$$\mathbf{F}(\hat{\boldsymbol{\xi}}) = -\rho\omega^2 \hat{\boldsymbol{\xi}} \,. \tag{25}$$

- For given equilibrium, collection of eigenvalues $\{\omega^2\}$ is *spectrum of ideal MHD.*

  ⇒ Generally both discrete and continuous (‘improper’) eigenvalues.

- The operator $\rho^{-1}\mathbf{F}$ is *self-adjoint* (for fixed boundary).

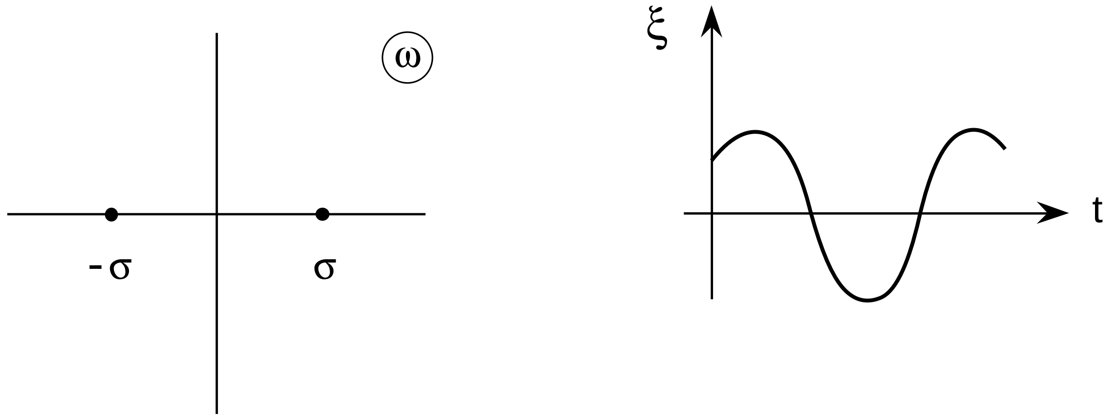
  ⇒ The eigenvalues ω^2 are real.

  ⇒ Same mathematical structure as for quantum mechanics!

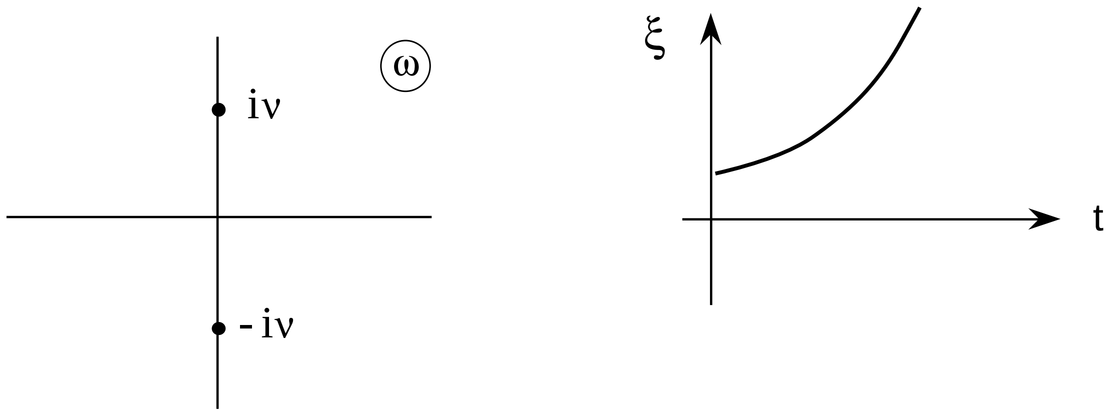
- Since ω^2 real, ω itself either real or purely imaginary

  $\Rightarrow$ In ideal MHD, only *stable waves* ($\omega^2 > 0$) òr *exponential instabilities* ($\omega^2 < 0$):

(a)



(b)

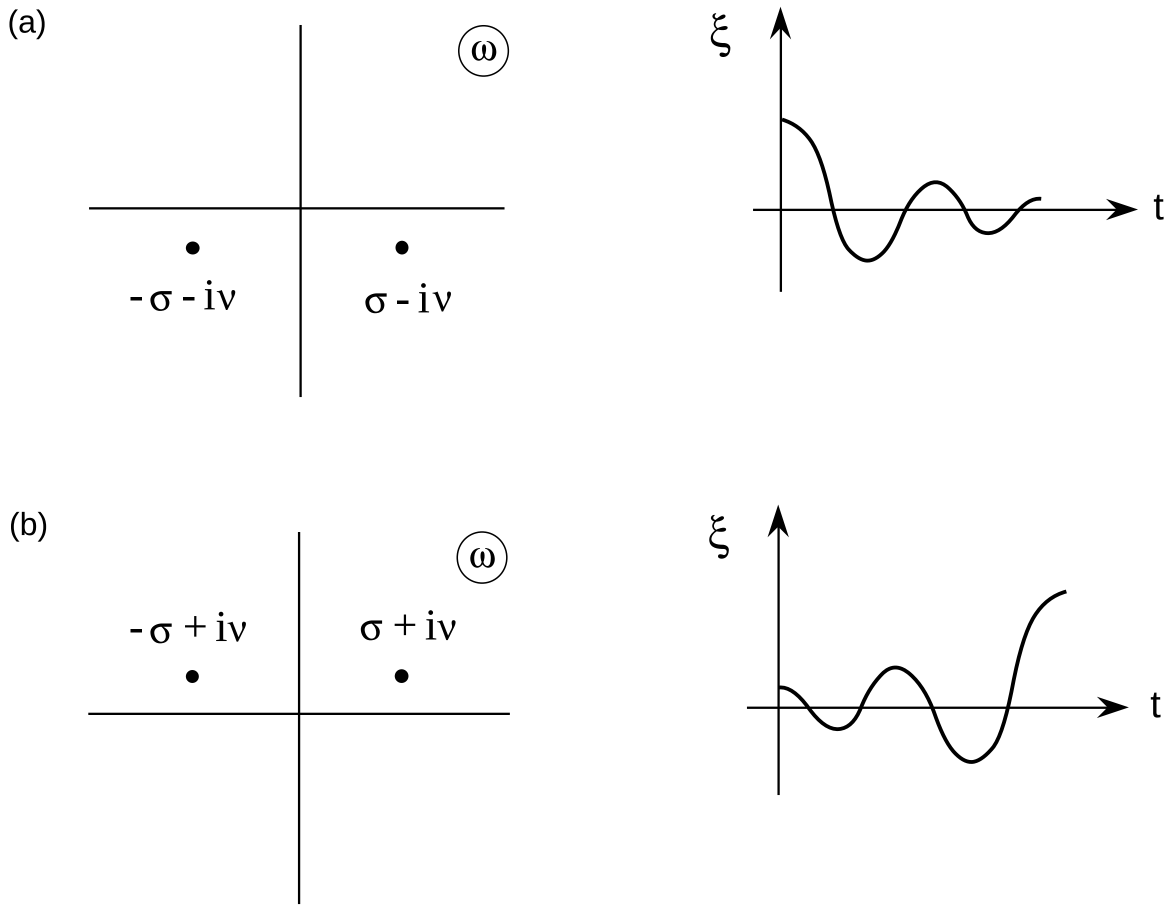


$\Rightarrow$ Crudely, $\mathbf{F}(\hat{\boldsymbol{\xi}}) \sim -\hat{\boldsymbol{\xi}}$ for $\omega^2 > 0$ and $\sim \hat{\boldsymbol{\xi}}$ for $\omega^2 < 0$ (cf. intuitive picture).

Dissipative MHD

- In resistive MHD, operators no longer self-adjoint, eigenvalues ω^2 complex.

  $\Rightarrow$ *Stable, damped waves* and *'overstable' modes ($\equiv$ instabilities):*



Stability in ideal MHD

- For ideal MHD, transition from stable to unstable through $\omega^2 = 0$: *marginal stability.*

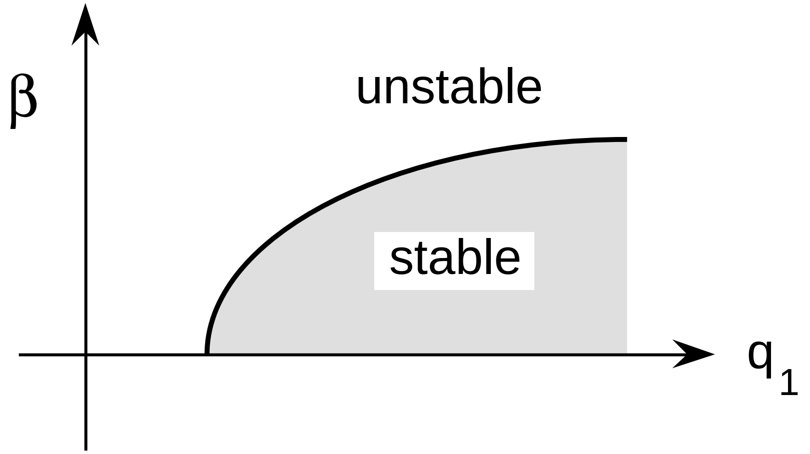
  $\Rightarrow$ Study marginal equation of motion

$$\mathbf{F}(\hat{\boldsymbol{\xi}}) = 0\,. \tag{26}$$

  $\Rightarrow$ In general, this equation has no solution since $\omega^2 = 0$ is not an eigenvalue.

- Can vary equilibrium parameters until zero eigenvalue is reached, e.g. in *tokamak stability analysis,* the parameters $\beta \equiv 2\mu_0 p/B^2$ and 'safety factor' $q_1 \sim 1/I_p$.

  $\Rightarrow$ Find critical curve along which $\omega^2 = 0$ is an eigenvalue:



  $\Rightarrow$ this curve separates stable from unstable parameter states.

Physical meaning of the terms of $\mathbf{F}$

- Rearrange terms:

$$\mathbf{F}(\boldsymbol{\xi}) = \nabla(\gamma p \nabla \cdot \boldsymbol{\xi}) - \mathbf{B} \times (\nabla \times \mathbf{Q}) + \nabla(\boldsymbol{\xi} \cdot \nabla p) + \mathbf{j} \times \mathbf{Q} + \nabla\Phi\, \nabla \cdot (\rho \boldsymbol{\xi})\,. \quad (27)$$

  First two terms (with γp and $\mathbf{B}$) present in *homogeneous equilibria*, last three terms only in *inhomogeneous equilibria* (when ∇p, $\mathbf{j}$, $\nabla\Phi \neq 0$).

- Hogeneous equilibria

  $\Rightarrow$ isotropic force $\nabla(\gamma p \nabla \cdot \boldsymbol{\xi})$: compressible sound waves;

  $\Rightarrow$ anisotropic force $\mathbf{B} \times (\nabla \times \mathbf{Q})$: field line bending Alfvén waves;

  $\Rightarrow$ waves always stable (see below).

- Inhomogeneous equilibria have *pressure gradients, currents, gravity*

  $\Rightarrow$ *potential sources for instability:* will require extensive study!

Homogeneous case

- Sound speed $c \equiv \sqrt{\gamma p/\rho}$ and Alfvén speed $\mathbf{b} \equiv \mathbf{B}/\sqrt{\rho}$ constant, so that

$$\rho^{-1}\mathbf{F}(\hat{\boldsymbol{\xi}}) = c^2\nabla\nabla\cdot\hat{\boldsymbol{\xi}} + \mathbf{b}\times(\nabla\times(\nabla\times(\mathbf{b}\times\hat{\boldsymbol{\xi}}))) = -\omega^2\hat{\boldsymbol{\xi}}\,. \tag{28}$$

  Plane wave solutions $\hat{\boldsymbol{\xi}} \sim \exp(i\mathbf{k}\cdot\mathbf{r})$ give

$$\rho^{-1}\mathbf{F}(\hat{\boldsymbol{\xi}}) = \big[-(\mathbf{k}\cdot\mathbf{b})^2\,\mathbf{I} - (b^2+c^2)\,\mathbf{kk} + \mathbf{k}\cdot\mathbf{b}\,(\mathbf{kb}+\mathbf{bk})\big]\cdot\hat{\boldsymbol{\xi}} = -\omega^2\hat{\boldsymbol{\xi}} \tag{29}$$

  $\Rightarrow$ recover the stable waves of Chapter 5.

- Recall: slow, Alfvén, fast eigenvectors $\hat{\boldsymbol{\xi}}_s$, $\hat{\boldsymbol{\xi}}_A$, $\hat{\boldsymbol{\xi}}_f$ form orthogonal triad

  $\Rightarrow$ can decompose any vector in combination of these 3 eigenvectors of $\mathbf{F}$;

  $\Rightarrow$ eigenvectors span whole space: *Hilbert space of plasma displacements.*

- Extract Alfvén wave (transverse incompressible $\mathbf{k}\cdot\boldsymbol{\xi} = 0$, $\mathbf{B}$ and $\mathbf{k}$ along z):

$$\rho^{-1}\hat{F}_y = b^2\,\frac{\partial^2\hat{\xi}_y}{\partial z^2} = -k_z^2 b^2\,\hat{\xi}_y = \frac{\partial^2\hat{\xi}_y}{\partial t^2} = -\omega^2\,\hat{\xi}_y\,, \tag{30}$$

  $\Rightarrow$ *Alfvén waves,* $\omega^2 = \omega_A^2 \equiv k_z^2 b^2$, *dynamical centerpiece of MHD spectral theory.*

Hilbert space

- Consider plasma volume V enclosed by wall W, with two displacement vector fields (satisfying the BCs):

$$\begin{aligned} \boldsymbol{\xi} &= \boldsymbol{\xi}(\mathbf{r},t) \quad \text{(on } V), \qquad \text{where} \quad \mathbf{n}\cdot\boldsymbol{\xi} = 0 \quad \text{(at } W), \\ \boldsymbol{\eta} &= \boldsymbol{\eta}(\mathbf{r},t) \quad \text{(on } V), \qquad \text{where} \quad \mathbf{n}\cdot\boldsymbol{\eta} = 0 \quad \text{(at } W). \end{aligned} \tag{31}$$

  Define inner product (weighted by the density):

$$\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle \equiv \tfrac{1}{2} \int \rho\, \boldsymbol{\xi}^* \cdot \boldsymbol{\eta}\, dV\,, \tag{32}$$

  and associated norm

$$\|\boldsymbol{\xi}\| \equiv \langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle^{1/2}\,. \tag{33}$$

- All functions with finite norm $\|\boldsymbol{\xi}\| < \infty$ form linear function space, a *Hilbert space.*

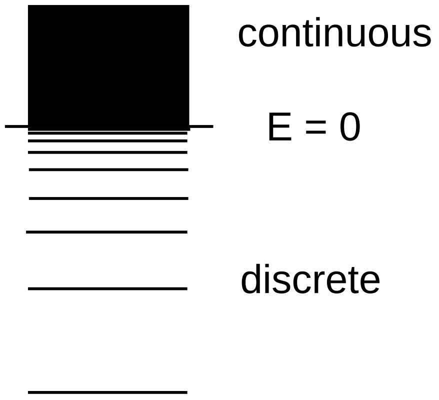
  $\Rightarrow$ Force operator $\mathbf{F}$ is *linear operator in Hilbert space* of vector displacements.

Analogy with quantum mechanics

- Recall Schrödinger equation for wave function ψ :

$$H\psi = E\psi \, . \tag{34}$$

$\Rightarrow$ Eigenvalue equation for Hamiltonian H with eigenvalues E (energy levels).



$\Rightarrow$ Spectrum of eigenvalues $\{E\}$ consists of discrete spectrum for bound states ($E < 0$) and continuous spectrum for free particle states ($E > 0$).

$\Rightarrow$ Norm $\|\psi\| \equiv \langle \psi, \psi \rangle^{1/2}$ gives probability to find particle in the volume.

- Central property in quantum mechanics: Hamiltonian H is *self-adjoint* linear operator in Hilbert space of wave functions,

$$\langle \psi_1, H\psi_2 \rangle = \langle H\psi_1, \psi_2 \rangle \, . \tag{35}$$

Back to MHD

- How about the force operator $\mathbf{F}$? Is it self-adjoint and, if so, what does it mean?

- Self-adjointness is related to energy conservation. For example, finite norm of $\boldsymbol{\xi}$, or its time derivative $\dot{\boldsymbol{\xi}}$, means that the kinetic energy is bounded:

$$K \equiv \tfrac{1}{2}\int \rho \mathbf{v}^2\, dV \approx \tfrac{1}{2}\int \rho \dot{\boldsymbol{\xi}}^2\, dV = \langle \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}} \rangle \equiv \|\dot{\boldsymbol{\xi}}\|^2 . \qquad (36)$$

  Consequently, the potential energy (related to $\mathbf{F}$, as we will see) is also bounded.

- The good news: *force operator $\rho^{-1}\mathbf{F}$ is self-adjoint linear operator* in Hilbert space of plasma displacement vectors:

$$\langle \boldsymbol{\eta}, \rho^{-1}\mathbf{F}(\boldsymbol{\xi})\rangle \equiv \tfrac{1}{2}\int \boldsymbol{\eta}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV = \tfrac{1}{2}\int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\eta}^*)\, dV \equiv \langle \rho^{-1}\mathbf{F}(\boldsymbol{\eta}), \boldsymbol{\xi}\rangle . \qquad (37)$$

  $\Rightarrow$ The mathematical analogy with quantum mechanics is complete.

- And the bad news: the proof of that central property is horrible!

Proving self-adjointness

- Proving

$$\int \boldsymbol{\eta}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV = \int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\eta}^*)\, dV$$

involves lots of tedious vector manipulations, with two returning ingredients:

  - use of equilibrium relations $\mathbf{j} \times \mathbf{B} = \nabla p + \rho \nabla \Phi$, $\mathbf{j} = \nabla \times \mathbf{B}$, $\nabla \cdot \mathbf{B} = 0$;
  - manipulation of volume integral to symmetric part in $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ and divergence term, which transforms into surface integral on which BCs are applied.

- Notational conveniences:

  - defining magnetic field perturbations associated with $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$,

$$\begin{aligned} \mathbf{Q}(\mathbf{r}) &\equiv \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) \qquad \text{(on } V\text{)}, \\ \mathbf{R}(\mathbf{r}) &\equiv \nabla \times (\boldsymbol{\eta} \times \mathbf{B}) \qquad \text{(on } V\text{)}; \end{aligned} \tag{38}$$

  - exploiting real-type scalar product,

$$\boldsymbol{\eta}^* \cdot \mathbf{F}(\boldsymbol{\xi}) + \text{complex conjugate} \quad \Rightarrow \quad \boldsymbol{\eta} \cdot \mathbf{F}(\boldsymbol{\xi}) .$$

- Omitting intermediate steps [see book: Sec. 6.2.3], we get useful, near-final result:

$$\int \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})\,dV = -\int \{\,\gamma p\,\nabla\cdot\boldsymbol{\xi}\,\nabla\cdot\boldsymbol{\eta} + \mathbf{Q}\cdot\mathbf{R} + \tfrac{1}{2}\nabla p\cdot(\boldsymbol{\xi}\,\nabla\cdot\boldsymbol{\eta} + \boldsymbol{\eta}\,\nabla\cdot\boldsymbol{\xi}) + \tfrac{1}{2}\mathbf{j}\cdot(\boldsymbol{\eta}\times\mathbf{Q} + \boldsymbol{\xi}\times\mathbf{R}) - \tfrac{1}{2}\nabla\Phi\cdot[\,\boldsymbol{\eta}\,\nabla\cdot(\rho\boldsymbol{\xi}) + \boldsymbol{\xi}\,\nabla\cdot(\rho\boldsymbol{\eta})\,]\,\}\,dV + \int \mathbf{n}\cdot\boldsymbol{\eta}\,[\,\gamma p\,\nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla p - \mathbf{B}\cdot\mathbf{Q}\,]\,dS\,. \qquad (39)$$

  This *symmetric* expression is general, valid for all model I–V problems.

- Restricting to model I (wall on the plasma), surface integrals vanish because of BC $\mathbf{n}\cdot\boldsymbol{\xi} = 0$, and **self-adjointness results:**

$$\int \{\boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi}) - \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\eta})\}\,dV = \int \{\,\mathbf{n}\cdot\boldsymbol{\eta}\,[\,\gamma p\,\nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla p - \mathbf{B}\cdot\mathbf{Q}\,] - \mathbf{n}\cdot\boldsymbol{\xi}\,[\,\gamma p\,\nabla\cdot\boldsymbol{\eta} + \boldsymbol{\eta}\cdot\nabla p - \mathbf{B}\cdot\mathbf{R}\,]\,\}\,dS = 0\,, \quad \textit{QED}\,. \qquad (40)$$

- Proof of self-adjointness for model II, etc. is rather straightforward now. It involves manipulating the surface term, using the pertinent BCs, to volume integral over the external vacuum region + again a vanishing surface integral over the wall.

Important result

- *The eigenvalues of $\rho^{-1}\mathbf{F}$ are real.*
- Proof
  - Consider pair of eigenfunction $\boldsymbol{\xi}_n$ and eigenvalue $-\omega_n^2$:

$$\rho^{-1}\mathbf{F}(\boldsymbol{\xi}_n) = -\omega_n^2\,\boldsymbol{\xi}_n\,;$$

  - take complex conjugate:

$$\rho^{-1}\mathbf{F}^*(\boldsymbol{\xi}_n) = \rho^{-1}\mathbf{F}(\boldsymbol{\xi}_n^*) = -\omega_n^{2*}\,\boldsymbol{\xi}_n^*\,;$$

  - multiply 1st equation with $\boldsymbol{\xi}_n^*$ and 2nd with $\boldsymbol{\xi}_n$, subtract, integrate over volume, and exploit self-adjointness:

$$0 = (\omega_n^2 - \omega_n^{2*})\,\|\boldsymbol{\xi}\|^2 \quad\Rightarrow\quad \omega_n^2 = \omega_n^{2*}\,, \quad \textit{QED}\,.$$

- Consequently, *ω^2 either ≥ 0 (stable) or < 0 (unstable):* everything falls in place!

Quadratic forms for potential energy

- Alternative representation is obtained from expressions for kinetic enery K and potential energy W, *exploiting energy conservation:* $H \equiv W + K = \text{const}\,.$

- (a) Use expression for K (already encountered) and equation of motion:

$$\frac{dK}{dt} \equiv \frac{d}{dt}\left[\tfrac{1}{2}\int \rho\,|\dot{\boldsymbol{\xi}}|^2\,dV\right] = \int \rho\,\dot{\boldsymbol{\xi}}^* \cdot \ddot{\boldsymbol{\xi}}\,dV = \int \dot{\boldsymbol{\xi}}^* \cdot \mathbf{F}(\boldsymbol{\xi})\,dV\,. \qquad (41)$$

(b) Exploit energy conservation and self-adjointness:

$$\frac{dW}{dt} = -\frac{dK}{dt} = -\tfrac{1}{2}\int\left[\dot{\boldsymbol{\xi}}^* \cdot \mathbf{F}(\boldsymbol{\xi}) + \boldsymbol{\xi}^* \cdot \mathbf{F}(\dot{\boldsymbol{\xi}})\right] dV = \frac{d}{dt}\left[-\tfrac{1}{2}\int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\,dV\right].$$

(c) Integration yields **linearized potential energy expression:**

$$W = -\tfrac{1}{2}\int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\,dV\,. \qquad (42)$$

- Intuitive meaning of W: potential energy increase from *work done against force* $\mathbf{F}$ (hence, minus sign), with $\frac{1}{2}$ since displacement builds up from 0 to final value.

- **More useful form of W** follows from earlier expression (39) (with $\boldsymbol{\eta} \rightarrow \boldsymbol{\xi}^*$) used in self-adjointness proof:

$$W = \tfrac{1}{2} \int \big[\, \gamma p \, |\nabla \cdot \boldsymbol{\xi}|^2 + |\mathbf{Q}|^2 + (\boldsymbol{\xi} \cdot \nabla p) \nabla \cdot \boldsymbol{\xi}^* + \mathbf{j} \cdot \boldsymbol{\xi}^* \times \mathbf{Q} - (\boldsymbol{\xi}^* \cdot \nabla \Phi) \, \nabla \cdot (\rho \boldsymbol{\xi}) \,\big] \, dV \,, \tag{43}$$

to be used with model I BC

$$\mathbf{n} \cdot \boldsymbol{\xi} = 0 \qquad \text{(at the wall)} \,. \tag{44}$$

- Earlier discussion on stability can now be completed:
  - first two terms (acoustic and magnetic energy) positive definite
    $\Rightarrow$ homogeneous plasma stable;
  - last three terms (pressure gradient, current, gravity) can have either sign
    $\Rightarrow$ *inhomogeneous plasma may be unstable* (requires extensive analysis).

Three variational principles

- Recall three levels of description with *differential equations:*

  (a) Equation of motion (20): $\mathbf{F}(\boldsymbol{\xi}) = \rho\, \ddot{\boldsymbol{\xi}}$ $\Rightarrow$ full dynamics;

  (b) Normal mode equaton (25): $\mathbf{F}(\hat{\boldsymbol{\xi}}) = -\rho\omega^2\hat{\boldsymbol{\xi}}$ $\Rightarrow$ spectrum of modes;

  (c) Marginal equation of motion (26): $\mathbf{F}(\hat{\boldsymbol{\xi}}) = 0$ $\Rightarrow$ stability only.

- Exploiting quadratic forms W and K yields *three variational counterparts:*

  (a) Hamilton's principle $\Rightarrow$ full dynamics;

  (b) Rayleigh–Ritz spectral principle $\Rightarrow$ spectrum of modes;

  (c) Energy principle $\Rightarrow$ stability only.

(a) Hamilton's principle

- Variational formulation of linear dynamics in terms of Lagrangian:

  The evolution of the system from time t_1 to time t_2 through the perturbation $\boldsymbol{\xi}(\mathbf{r}, t)$ is such that the variation of the integral of the Lagrangian vanishes,

$$\delta \int_{t_1}^{t_2} L\, dt = 0\,, \qquad L \equiv K - W\,, \tag{45}$$

  with

$$K = K[\dot{\boldsymbol{\xi}}] = \tfrac{1}{2}\int \rho\, \dot{\boldsymbol{\xi}}^* \cdot \dot{\boldsymbol{\xi}}\, dV\,,$$

$$W = W[\boldsymbol{\xi}] = -\tfrac{1}{2}\int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV\,.$$

- Minimization (see Goldstein on classical fields) gives Euler–Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\xi}_j} + \sum_k \frac{d}{dx_k}\frac{\partial \mathcal{L}}{\partial(\partial \xi_j/\partial x_k)} - \frac{\partial \mathcal{L}}{\partial \xi_j} = 0 \quad \Rightarrow \quad \mathbf{F}(\boldsymbol{\xi}) = \rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2}\,, \tag{46}$$

  which is the equation of motion, QED.

(b) Rayleigh–Ritz spectral principle

- Consider quadratic forms W and K (here I) *for normal modes* $\hat{\boldsymbol{\xi}}\,\mathrm{e}^{-\mathrm{i}\omega t}$ *:*

$$\mathbf{F}(\hat{\boldsymbol{\xi}}) = -\rho\omega^2\hat{\boldsymbol{\xi}} \quad \Rightarrow \quad \underbrace{-\tfrac{1}{2}\int \hat{\boldsymbol{\xi}}^* \cdot \mathbf{F}(\hat{\boldsymbol{\xi}})\, dV}_{\equiv\, W[\hat{\boldsymbol{\xi}}]} = \omega^2 \cdot \underbrace{\tfrac{1}{2}\int \rho\hat{\boldsymbol{\xi}}^* \cdot \hat{\boldsymbol{\xi}}\, dV}_{\equiv\, I[\hat{\boldsymbol{\xi}}]} \,.$$

  This gives

$$\omega^2 = \frac{W[\hat{\boldsymbol{\xi}}]}{I[\hat{\boldsymbol{\xi}}]} \qquad \text{for normal modes}\,. \tag{47}$$

  True, but useless: just conclusion a posteriori on $\boldsymbol{\xi}$ and ω^2, no recipe to find them.

- Obtain recipe by turning this into Rayleigh–Ritz variational expression for eigenvalues:

  Eigenfunctions $\boldsymbol{\xi}$ of the operator $\rho^{-1}\mathbf{F}$ make the Rayleigh quotient

$$\Lambda[\boldsymbol{\xi}] \equiv \frac{W[\boldsymbol{\xi}]}{I[\boldsymbol{\xi}]} \tag{48}$$

  stationary; eigenvalues ω^2 are the stationary values of Λ.

  $\Rightarrow$ Practical use: approximate eigenvalues/eigenfunctions by minimizing Λ over linear combination of pre-chosen set of trial functions $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \ldots \boldsymbol{\eta}_N)$.

(c) Energy principle for stability

- Since $I \equiv \|\boldsymbol{\xi}\|^2 \geq 0$, Rayleigh–Ritz variational principle offers possibility of testing for stability by *inserting trial functions in W:*
  - If $W[\boldsymbol{\xi}] < 0$ for single $\boldsymbol{\xi}$, at least one eigenvalue $\omega^2 < 0$ and system is *unstable*;
  - If $W[\boldsymbol{\xi}] > 0$ for all $\boldsymbol{\xi}$s, eigenvalues $\omega^2 < 0$ do not exist and system is *stable*.

- $\Rightarrow$ **Energy principle:** *An equilibrium is stable if (sufficient) and only if (necessary)*

$$W[\boldsymbol{\xi}] > 0 \tag{49}$$

  for all displacements $\boldsymbol{\xi}(\mathbf{r})$ that are bound in norm and satisfy the BCs.

- Summarizing, the variational approach offers three methods to determine stability:
  (1) Guess a trial function $\boldsymbol{\xi}(\mathbf{r})$ such that $W[\boldsymbol{\xi}] < 0$ for a certain system
  $\Rightarrow$ *necessary stability ($\equiv$ sufficient instability) criterium*;
  (2) Investigate sign of W with complete set of arbitrarily normalized trial functions
  $\Rightarrow$ *necessary + sufficient stability criterium*;
  (3) Minimize W with complete set of properly normalized functions (i.e. with $I[\boldsymbol{\xi}]$, related to kinetic energy) $\Rightarrow$ *complete spectrum of (discrete) eigenvalues.*

Returning to the two viewpoints

- Spectral theory elucidates analogies between different parts of physics:

MHD		*Linear analysis*		*QM*
Force operator	$\Longleftrightarrow$	**Differential equations**	$\Longleftrightarrow$	**Schrödinger picture**
Energy principle	$\Longleftrightarrow$	**Quadratic forms**	$\Longleftrightarrow$	**Heisenberg picture**

  The analogy is through mathematics $\Uparrow$, not through physics!

- Linear operators in Hilbert space as such have nothing to do with quantum mechanics. Mathematical formulation by Hilbert (1912) preceded it by more than a decade. Essentially, the two 'pictures' are just translation to physics of *generalization of linear algebra to infinite-dimensional vector spaces* (Moser, 1973).

- Whereas quantum mechanics applies to rich arsenal of spherically symmetric systems (symmetry with respect to rotation groups), in MHD the constraint $\nabla \cdot \mathbf{B} = 0$ forbids spherical symmetry and implies much less obvious symmetries.
  $\Rightarrow$ Application of group theory to MHD is still in its infancy.

Two ‘pictures’ of MHD spectral theory:

Differential eqs. ('Schrödinger')	**Quadratic forms** ('Heisenberg')	
Equation of motion: $\mathbf{F}(\boldsymbol{\xi}) = \rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2}$	*Hamilton's principle:* $\delta \int_{t_1}^{t_2} \Big(K[\dot{\boldsymbol{\xi}}] - W[\boldsymbol{\xi}] \Big)\, dt = 0 \quad \Rightarrow$	Full dynamics: $\boldsymbol{\xi}(\mathbf{r}, t)$
Eigenvalue problem: $\mathbf{F}(\boldsymbol{\xi}) = -\rho\omega^2 \boldsymbol{\xi}$	*Rayleigh's principle:* $\delta \frac{W[\boldsymbol{\xi}]}{I[\boldsymbol{\xi}]} = 0 \quad \Rightarrow$	Spectrum $\{\omega^2\}$ & eigenf. $\{\boldsymbol{\xi}(\mathbf{r})\}$
Marginal equation: $\mathbf{F}(\boldsymbol{\xi}) = 0$	*Energy principle:* $W[\boldsymbol{\xi}] \gtrless 0 \quad \Rightarrow$	Stability $\binom{\mathrm{y}}{\mathrm{n}}$ & trial $\boldsymbol{\xi}(\mathbf{r})$

Why does the water fall out of the glass?

- Apply spectral theory and energy principle to simple fluid (no magnetic field) with *varying density in external gravitational field.* Equilibrium: $\nabla p = -\rho \nabla \Phi = \rho \mathbf{g}$.

$$W^f = \tfrac{1}{2} \int \big[\, \gamma p \, |\nabla \cdot \boldsymbol{\xi}|^2 + (\boldsymbol{\xi} \cdot \nabla p) \nabla \cdot \boldsymbol{\xi}^* - (\boldsymbol{\xi}^* \cdot \nabla \Phi) \, \nabla \cdot (\rho \boldsymbol{\xi}) \,\big] \, dV$$

$$= \tfrac{1}{2} \int \big[\, \gamma p \, |\nabla \cdot \boldsymbol{\xi}|^2 + \rho \mathbf{g} \cdot (\boldsymbol{\xi} \, \nabla \cdot \boldsymbol{\xi}^* + \boldsymbol{\xi}^* \nabla \cdot \boldsymbol{\xi}) + \mathbf{g} \cdot \boldsymbol{\xi}^* (\nabla \rho) \cdot \boldsymbol{\xi} \,\big] \, dV. \qquad (50)$$

Without gravity, fluid is stable since only positive definite first term remains.

- *Plane slab,* $p(x)$, $\rho(x)$, $\mathbf{g} = -g \mathbf{e}_x \;\Rightarrow\;$ equilibrium: $p' = -\rho g$.

$$W^f = \tfrac{1}{2} \int \big[\, \gamma p \, |\nabla \cdot \boldsymbol{\xi}|^2 - \rho g (\xi_x \nabla \cdot \boldsymbol{\xi}^* + \xi_x^* \nabla \cdot \boldsymbol{\xi}) - \rho' g |\xi_x|^2 \,\big] \, dV \, . \qquad (51)$$

- Energy principle according to method (1) illustrated by exploiting incompressible trial functions, $\nabla \cdot \boldsymbol{\xi} = 0$:

$$W^f = -\tfrac{1}{2} \int \rho' g |\xi_x|^2 \, dV \geq 0 \quad \Rightarrow \quad \rho' g \leq 0 \quad \text{(everywhere)} \, . \qquad (52)$$

$\Rightarrow$ *Necessary stability criterion:* **lighter fluid should be on top of heavier fluid.**

- Much sharper stability condition from energy principle according to method (2), where all modes (also compressible ones) are considered. Rearrange terms in Eq. (51):

$$W^f = \tfrac{1}{2} \int \Big[\gamma p \, \Big| \nabla \cdot \boldsymbol{\xi} - \frac{\rho g}{\gamma p} \xi_x \Big|^2 - \Big(\rho' g + \frac{\rho^2 g^2}{\gamma p} \Big) |\xi_x|^2 \Big] \, dV \, . \tag{53}$$

Since ξ_y and ξ_z only appear in $\nabla \cdot \boldsymbol{\xi}$, minimization with respect to them is trivial:

$$\nabla \cdot \boldsymbol{\xi} = \frac{\rho g}{\gamma p} \xi_x \, . \tag{54}$$

$\Rightarrow$ *Necessary and sufficient stability criterion:*

$$\rho' g + \frac{\rho^2 g^2}{\gamma p} \leq 0 \quad \text{(everywhere)} \, . \tag{55}$$

- Actually, we have now derived conditions for stability with respect to *internal modes*. Original water-air system requires extended energy principle with two-fluid interface (model II*), permitting description of *external modes*: our next subject. Physics will be the same: density gradient becomes density jump, that should be negative at the interface (light fluid above) for stability.

Interfaces

- So far, plasmas bounded by rigid wall (model I). Most applications require interface:
  - In tokamaks, very low density close to wall (created by ‘limiter’) is effectively vacuum $\Rightarrow$ *plasma–vacuum system* (model II);
  - In astrophysics, frequently density jump (e.g. to low-density force-free plasma) $\Rightarrow$ *plasma–plasma system* (model II*).

- Model II: split vacuum magnetic field in equilibrium part $\hat{\mathbf{B}}$ and perturbation $\hat{\mathbf{Q}}$.

Equilibrium: $\nabla \times \hat{\mathbf{B}} = 0\,, \;\; \nabla \cdot \hat{\mathbf{B}} = 0$, with BCs

$$\mathbf{n} \cdot \mathbf{B} = \mathbf{n} \cdot \hat{\mathbf{B}} = 0\,, \qquad [\![p + \tfrac{1}{2}B^2]\!] = 0 \quad \textit{(at interface } S\textit{)}\,, \tag{56}$$

$$\mathbf{n} \cdot \hat{\mathbf{B}} = 0 \qquad \textit{(at outer wall } \hat{W}\textit{)}\,. \tag{57}$$

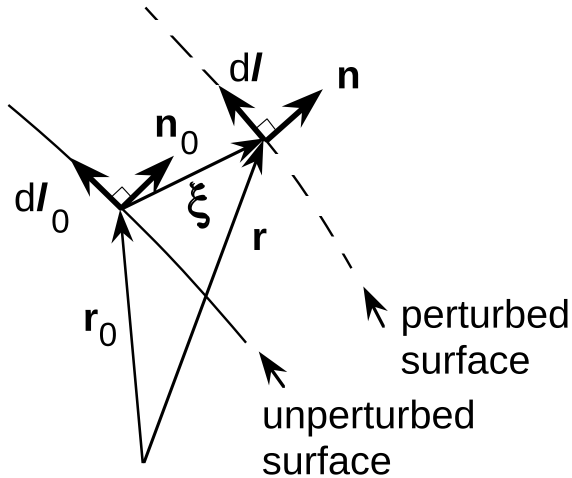
Perturbations: $\nabla \times \hat{\mathbf{Q}} = 0\,, \;\; \nabla \cdot \hat{\mathbf{Q}} = 0$, with two non-trivial BCs connecting $\hat{\mathbf{Q}}$ to the plasma variable $\boldsymbol{\xi}$ at the interface, and one BC at the wall:

$$\text{1st interface cond.}\,, \qquad \text{2nd interface cond.} \quad \textit{(at interface } S\textit{)}\,, \tag{58}$$

$$\mathbf{n} \cdot \hat{\mathbf{Q}} = 0 \qquad \textit{(at outer wall } \hat{W}\textit{)}\,. \tag{59}$$

Explicit derivation of interface conditions (58) below: Eqs. (62) and (63).

Boundary conditions for interface plasmas



- Need expression for *perturbation of the normal* $\mathbf{n}$ *to the interface.*
- Integrating Lagrangian time derivative of line element (see Chap. 4) yields:

$$\mathrm{D}(d\boldsymbol{l})/\mathrm{D}t = d\boldsymbol{l} \cdot (\nabla \mathbf{v}) \;\Rightarrow\; d\boldsymbol{l} \approx d\boldsymbol{l}_0 \cdot (\mathsf{I} + \nabla \boldsymbol{\xi})\,.$$

- For $d\boldsymbol{l}$ lying in the boundary surface:

$$0 = \mathbf{n} \cdot d\boldsymbol{l} \approx d\boldsymbol{l}_0 \cdot (\mathsf{I} + \nabla \boldsymbol{\xi}) \cdot (\mathbf{n}_0 + \mathbf{n}_{1\mathrm{L}}) \approx d\boldsymbol{l}_0 \cdot [\,(\nabla \boldsymbol{\xi}) \cdot \mathbf{n}_0 + \mathbf{n}_{1\mathrm{L}}\,]\,.$$

$\Rightarrow$ Lagrangian perturbation: $\mathbf{n}_{1\mathrm{L}} = -(\nabla \boldsymbol{\xi}) \cdot \mathbf{n}_0 + \boldsymbol{\lambda}$, with vector $\boldsymbol{\lambda} \perp d\boldsymbol{l}_0$.

Since $d\boldsymbol{l}_0$ has arbitrary direction in unperturbed surface, $\boldsymbol{\lambda}$ must be $\parallel \mathbf{n}_0$: $\boldsymbol{\lambda} = \mu \mathbf{n}_0$.
Since $|\mathbf{n}| = |\mathbf{n}_0| = 1$, we have $\mathbf{n}_0 \cdot \mathbf{n}_{1\mathrm{L}} = 0$, so that $\mu = \mathbf{n}_0 \cdot (\nabla \boldsymbol{\xi}) \cdot \mathbf{n}_0$.

This provides the *Lagrangian perturbation of the normal*:

$$\mathbf{n}_{1\mathrm{L}} = -(\nabla \boldsymbol{\xi}) \cdot \mathbf{n}_0 + \mathbf{n}_0\, \mathbf{n}_0 \cdot (\nabla \boldsymbol{\xi}) \cdot \mathbf{n}_0 = \mathbf{n}_0 \times \{\mathbf{n}_0 \times [\,(\nabla \boldsymbol{\xi}) \cdot \mathbf{n}_0\,]\}\,. \qquad (60)$$

- Original BCs for model II come from jump conditions of Chap. 4:

$$\text{(a)} \quad \mathbf{n}\cdot\mathbf{B} = \mathbf{n}\cdot\hat{\mathbf{B}} = 0 \qquad \textit{(at plasma–vacuum interface)}\,,$$

$$\text{(b)} \quad [\![p + \tfrac{1}{2}B^2]\!] = 0 \qquad \textit{(at plasma–vacuum interface)}\,.$$

Also need *Lagrangian perturbation of magnetic field* $\mathbf{B}$ *and pressure* p *at perturbed boundary position* $\mathbf{r}$, evaluated to first order:

$$\begin{aligned} \mathbf{B}|_{\mathbf{r}} &\approx (\mathbf{B}_0 + \mathbf{Q} + \boldsymbol{\xi}\cdot\nabla\mathbf{B}_0)|_{\mathbf{r}_0}\,, \\ p|_{\mathbf{r}} &\approx (p_0 + \pi + \boldsymbol{\xi}\cdot\nabla p_0)|_{\mathbf{r}_0} = (p_0 - \gamma p_0 \nabla\cdot\boldsymbol{\xi})|_{\mathbf{r}_0}\,. \end{aligned} \qquad (61)$$

- Insert Eqs. (60) and (61) into first part of above BC (a):

$$\begin{aligned} 0 \;=\; \mathbf{n}\cdot\mathbf{B} &\approx [\,\mathbf{n}_0 - (\nabla\boldsymbol{\xi})\cdot\mathbf{n}_0 + \mathbf{n}_0\,\mathbf{n}_0\cdot(\nabla\boldsymbol{\xi})\cdot\mathbf{n}_0\,]\cdot(\mathbf{B}_0 + \mathbf{Q} + \boldsymbol{\xi}\cdot\nabla\mathbf{B}_0) \\ &\approx -\mathbf{B}_0\cdot(\nabla\boldsymbol{\xi})\cdot\mathbf{n}_0 + \mathbf{n}_0\cdot\mathbf{Q} + \boldsymbol{\xi}\cdot(\nabla\mathbf{B}_0)\cdot\mathbf{n}_0 = -\mathbf{n}_0\cdot\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0) + \mathbf{n}_0\cdot\mathbf{Q}\,. \end{aligned}$$

Automatically satisfied since $\mathbf{Q} \equiv \nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)$. However, same derivation for second part of BC (a) gives *1st interface condition* relating $\boldsymbol{\xi}$ and $\hat{\mathbf{Q}}$:

$$\mathbf{n}\cdot\nabla\times(\boldsymbol{\xi}\times\hat{\mathbf{B}}) = \mathbf{n}\cdot\hat{\mathbf{Q}} \qquad \textit{(at plasma–vacuum interface } S\textit{)}\,. \qquad (62)$$

- Inserting Eqs. (61) into BC (b) yields *2nd interface condition* relating $\boldsymbol{\xi}$ and $\hat{\mathbf{Q}}$:

$$-\gamma p\nabla\cdot\boldsymbol{\xi} + \mathbf{B}\cdot\mathbf{Q} + \boldsymbol{\xi}\cdot\nabla(\tfrac{1}{2}B^2) = \hat{\mathbf{B}}\cdot\hat{\mathbf{Q}} + \boldsymbol{\xi}\cdot\nabla(\tfrac{1}{2}\hat{B}^2) \qquad \textit{(at } S\textit{)}\,. \qquad (63)$$

Extended energy principle

- Proof *self-adjointness* continues from integral (39) for $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, connected with vacuum 'extensions' $\hat{\mathbf{Q}}$, $\hat{\mathbf{R}}$ through BCs (59), (62), (63), giving symmetric quadratic form.
- Putting $\boldsymbol{\eta} = \boldsymbol{\xi}^*$, $\hat{\mathbf{R}} = \hat{\mathbf{Q}}^*$ in integrals gives **potential energy for interface plasmas:**

$$W[\boldsymbol{\xi}, \hat{\mathbf{Q}}] = -\tfrac{1}{2} \int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})\, dV = W^p[\boldsymbol{\xi}] + W^s[\xi_n] + W^v[\hat{\mathbf{Q}}]\,, \tag{64}$$

where

$$W^p[\boldsymbol{\xi}] = \tfrac{1}{2} \int \big[\, \gamma p\, |\nabla \cdot \boldsymbol{\xi}|^2 + |\mathbf{Q}|^2 + (\boldsymbol{\xi} \cdot \nabla p)\, \nabla \cdot \boldsymbol{\xi}^* + \mathbf{j} \cdot \boldsymbol{\xi}^* \times \mathbf{Q} - (\boldsymbol{\xi}^* \cdot \nabla \Phi)\, \nabla \cdot (\rho \boldsymbol{\xi})\, \big]\, dV\,, \tag{65}$$

$$W^s[\xi_n] = \tfrac{1}{2} \int |\mathbf{n} \cdot \boldsymbol{\xi}|^2\, \mathbf{n} \cdot [\nabla (p + \tfrac{1}{2} B^2)]\, dS\,, \tag{66}$$

$$W^v[\hat{\mathbf{Q}}] = \tfrac{1}{2} \int |\hat{\mathbf{Q}}|^2\, d\hat{V}\,. \tag{67}$$

Work against force $\mathbf{F}$ now leads to increase of potential energy of the plasma, W^p, of the plasma–vacuum surface, W^s, and of the vacuum, W^v.

- Variables $\boldsymbol{\xi}$ and $\hat{\mathbf{Q}}$ have to satisfy **essential boundary conditions:**

1) $\boldsymbol{\xi}$ regular on plasma volume V , (68)

2) $\mathbf{n} \cdot \nabla \times (\boldsymbol{\xi} \times \hat{\mathbf{B}}) = \mathbf{n} \cdot \hat{\mathbf{Q}}$ *(1st interface condition on S)* , (69)

3) $\mathbf{n} \cdot \hat{\mathbf{Q}} = 0$ *(on outer wall $\hat{W}$)* . (70)

- Note: Differential equations for $\hat{\mathbf{Q}}$ and 2nd interface condition need not be imposed ! They are absorbed in form of $W[\boldsymbol{\xi}, \hat{\mathbf{Q}}]$ and automatically satisfied upon minimization. For that reason 2nd interface condition (63) is called *natural boundary condition*.

- Great simplification by assuming *incompressible perturbations,* $\nabla \cdot \boldsymbol{\xi} = 0$:

$$W^p_{\rm inc}[\boldsymbol{\xi}] = \tfrac{1}{2} \int \left[|\mathbf{Q}|^2 + \mathbf{j} \cdot \boldsymbol{\xi}^* \times \mathbf{Q} - (\boldsymbol{\xi}^* \cdot \nabla \Phi)\, \nabla \rho \cdot \boldsymbol{\xi} \right] dV \,. \qquad (71)$$

Note: In equation of motion, one cannot simply put $\nabla \cdot \boldsymbol{\xi} = 0$ and drop $-\gamma p \nabla \cdot \boldsymbol{\xi}$ from pressure perturbation π, since that leads to overdetermined system of equations for 3 components of $\boldsymbol{\xi}$. Consistent procedure: apply two limits $\gamma \to \infty$ and $\nabla \cdot \boldsymbol{\xi} \to 0$ simultaneously such that Lagrangian perturbation $\pi_{\rm L} \equiv -\gamma p \nabla \cdot \boldsymbol{\xi}$ remains finite.

Application to Rayleigh–Taylor instability

- Apply extended energy principle to **gravitational instability of magnetized plasma supported from below by vacuum magnetic field:** Model problem for plasma confinement with clear separation of inner plasma and outer vacuum, and instabilities localized at interface (free-boundary or surface instabilities). Rayleigh–Taylor instability of magnetized plasmas involves the basic concepts of *interchange instability*, *magnetic shear stabilization*, and *wall stabilization*. These instabilities arise in wide class of astrophysical situations, e.g. *Parker instability* in galactic plasmas.

- Gravitational equilibrium in magnetized plasma:

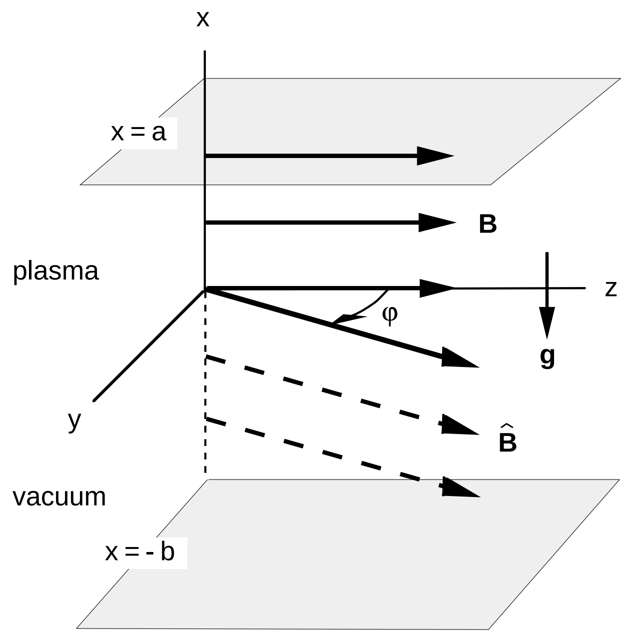
$$\rho = \rho_0 \,, \quad \mathbf{B} = B_0 \mathbf{e}_z \,, \quad p = p_0 - \rho_0 g x \,, \qquad (72)$$

pressure balance at plasma–vacuum interface:

$$p_0 + \tfrac{1}{2} B_0^2 = \tfrac{1}{2} \hat{B}_0^2 \,, \qquad (73)$$

vacuum magnetic field:

$$\hat{\mathbf{B}} = \hat{B}_0 (\sin\varphi \, \mathbf{e}_y + \cos\varphi \, \mathbf{e}_z) \,. \qquad (74)$$



- Insert equilibrium into $W^p_{\rm inc}$, W^s, W^v, where jump in surface integral (66) gives driving term of the gravitational instability:

$$\mathbf{n} \cdot [\![\nabla (p + \tfrac{1}{2}B^2)]\!] = p' = -\rho_0 g \,. \tag{75}$$

Potential energy $W[\boldsymbol{\xi}, \hat{\mathbf{Q}}]$ becomes:

$$W^p = \tfrac{1}{2} \int |\mathbf{Q}|^2 \, dV \,, \qquad \mathbf{Q} \equiv \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) \,, \qquad \nabla \cdot \boldsymbol{\xi} = 0 \,, \tag{76}$$

$$W^s = -\tfrac{1}{2}\rho_0 g \int |\mathbf{n} \cdot \boldsymbol{\xi}|^2 \, dS \,, \tag{77}$$

$$W^v = \tfrac{1}{2} \int |\hat{\mathbf{Q}}|^2 \, d\hat{V} \,, \qquad \nabla \cdot \hat{\mathbf{Q}} = 0 \,. \tag{78}$$

Task: *Minimize $W[\boldsymbol{\xi}, \hat{\mathbf{Q}}]$ for divergence-free trial functions $\boldsymbol{\xi}$ and $\hat{\mathbf{Q}}$ that satisfy the essential boundary conditions (68)–(70).*

- Slab is translation symmetric in y and z $\Rightarrow$ *Fourier modes* do not couple:

$$\boldsymbol{\xi} = \big(\xi_x(x), \xi_y(x), \xi_z(x)\big)\, e^{i(k_y y + k_z z)} \,, \quad \text{similarly for } \hat{\mathbf{Q}} \,. \tag{79}$$

- Eliminating ξ_z from W^p, and $\hat{Q}_z$ from W^v, by using $\nabla \cdot \boldsymbol{\xi} = 0$ and $\nabla \cdot \hat{\mathbf{Q}} = 0$, yields *1D expressions:*

$$W^p = \tfrac{1}{2} B_0^2 \int_0^a \big[\, k_z^2 (|\xi_x|^2 + |\xi_y|^2) + |\xi_x' + i k_y \xi_y|^2 \,\big]\, dx \,, \tag{80}$$

$$W^s = -\tfrac{1}{2} \rho_0 g |\xi_x(0)|^2 \,, \tag{81}$$

$$W^v = \tfrac{1}{2} \int_{-b}^0 \Big[\, |\hat{Q}_x|^2 + |\hat{Q}_y|^2 + \frac{1}{k_z^2} |\hat{Q}_x' + i k_y \hat{Q}_y|^2 \,\Big]\, dx \,. \tag{82}$$

- To be minimized subject to normalization that may be chosen freely *for stability:*

$$\xi_x(0) = \text{const} \,, \tag{83}$$

or full physical norm if we wish to obtain *growth rate of instabilities:*

$$I = \tfrac{1}{2} \rho_0 \int_0^a \Big[\, |\xi_x|^2 + |\xi_y|^2 + \frac{1}{k_z^2} |\xi_x' + i k_y \xi_y|^2 \,\Big]\, dx \,. \tag{84}$$

- Essential boundary conditions always need to be satisfied:

$$\xi_x(a) = 0 \,, \tag{85}$$

$$\hat{Q}_x(0) = i \mathbf{k}_0 \cdot \hat{\mathbf{B}}\, \xi_x(0) \,, \qquad \mathbf{k}_0 \equiv (0, k_y, k_z) \,, \tag{86}$$

$$\hat{Q}_x(-b) = 0 \,. \tag{87}$$

Stability analysis

- Minimization with respect to ξ_y and $\hat{Q}_y$ only involves minimization of W^p and W^v:

$$W^p = \tfrac{1}{2}B_0^2 \int_0^a \big[\, \frac{k_z^2}{k_0^2}{\xi'_x}^2 + k_z^2\xi_x^2 + \Big|\frac{k_y}{k_0}\xi'_x + ik_0\xi_y\Big|^2 \,\big]\, dx = \tfrac{1}{2}k_z^2 B_0^2 \int_0^a \big(\frac{1}{k_0^2}\,{\xi'_x}^2 + \xi_x^2\big)\, dx\,,$$

$$W^v = \tfrac{1}{2}\int_{-b}^0 \big[\, |\hat{Q}_x|^2 + \frac{1}{k_0^2}|\hat{Q}'_x|^2 \,\big] + \frac{1}{k_z^2}\Big|\frac{k_y}{k_0}\hat{Q}'_x + ik_0\hat{Q}_y\Big|^2 \,\big]\, dx = \tfrac{1}{2}\int_{-b}^0 \big(\frac{1}{k_0^2}|\hat{Q}'_x|^2 + |\hat{Q}_x|^2\big)\, dx.$$

$\Rightarrow$ Determine $\xi_x(x)$ and $\hat{Q}_x(x)$, joined by 1st interface condition (86) at $x = 0$.

- Recall *variational analysis:* Minimization of quadratic form

$$W[\xi] = \tfrac{1}{2}\int_0^a (F\xi'^2 + G\xi^2)\, dx = \tfrac{1}{2}\big[\, F\xi\xi' \,\big]_0^a - \tfrac{1}{2}\int_0^a \big[\,(F\xi')' - G\xi\,\big]\xi\, dx \qquad (88)$$

is effected by variation $\delta\xi(x)$ of the unknown function $\xi(x)$:

$$\delta W = \int_0^a \big(F\xi'\delta\xi' + G\xi\delta\xi\big)\, dx = \big[\, F\xi'\delta\xi \,\big]_0^a - \int_0^a \big[\,(F\xi')' - G\xi\,\big]\delta\xi\, dx = 0\,. \qquad (89)$$

Since $\delta\xi = 0$ at boundaries, solution of *Euler–Lagrange equation* minimizes W:

$$(F\xi')' - G\xi = 0 \quad \Rightarrow \quad W_{\min} = \tfrac{1}{2}\big[\, F\xi\xi' \,\big]_0^a = -\tfrac{1}{2}[F\xi\xi'](x{=}0)\,, \qquad (90)$$

where we imposed upper wall BC $\xi(a) = 0$, appropriate for our application.

- Minimization of integrals W^p and W^v yields following Euler–Lagrange equations, with solutions satisfying BCs on upper and lower walls:

$$\xi_x'' - k_0^2 \xi_x = 0 \qquad \Rightarrow \qquad \xi_x = C \sinh\left[k_0(a - x)\right],$$
$$\hat{Q}_x'' - k_0^2 \hat{Q}_x = 0 \qquad \Rightarrow \qquad \hat{Q}_x = i\hat{C} \sinh\left[k_0(x + b)\right]. \tag{91}$$

Modes are *wave-like* in horizontal, but *evanescent* in vertical direction.

- C and $\hat{C}$ determined by normalization (83) and 1st interface condition (86):

$$\hat{C} \sinh(k_0 b) = \mathbf{k}_0 \cdot \hat{\mathbf{B}}\, \xi_x(0) = C \mathbf{k}_0 \cdot \hat{\mathbf{B}} \sinh(k_0 a)\,. \tag{92}$$

- Inserting solutions of Euler–Lagrange equations back into energy integrals, yields final expression for W in terms of *constant boundary contributions at $x = 0$:*

$$W = -\frac{k_z^2 B_0^2}{2k_0^2}\, \xi_x(0) \xi_x'(0) - \tfrac{1}{2} \rho_0 g\, \xi_x^2(0) + \frac{1}{2k_0^2} \,|\hat{Q}_x(0) \hat{Q}_x'(0)|$$
$$= \frac{\xi_x^2(0)}{2k_0 \tanh(k_0 a)} \left[(\mathbf{k}_0 \cdot \mathbf{B})^2 - \rho_0 k_0 g \tanh(k_0 a) + (\mathbf{k}_0 \cdot \hat{\mathbf{B}})^2 \frac{\tanh(k_0 a)}{\tanh(k_0 b)} \right]. \tag{93}$$

Expression inside square brackets corresponds to growth rate.

Growth rate

- With full norm (84), we obtain *dispersion equation of the Rayleigh–Taylor instability:*

$$\omega^2 = \frac{W}{I} = \frac{1}{\rho_0}\left[(\mathbf{k}_0 \cdot \mathbf{B})^2 - \rho_0 k_0 g \, \tanh(k_0 a) + (\mathbf{k}_0 \cdot \hat{\mathbf{B}})^2 \frac{\tanh(k_0 a)}{\tanh(k_0 b)} \right] . \qquad (94)$$

- *Field line bending energies* $\sim \frac{1}{2}(\mathbf{k}_0 \cdot \mathbf{B})^2$ for plasma and $\sim \frac{1}{2}(\mathbf{k}_0 \cdot \hat{\mathbf{B}})^2$ for vacuum, *destabilizing gravitational energy* $\sim -\frac{1}{2}\rho_0 k_0 g \tanh(k_0 a)$ due to motion interface.

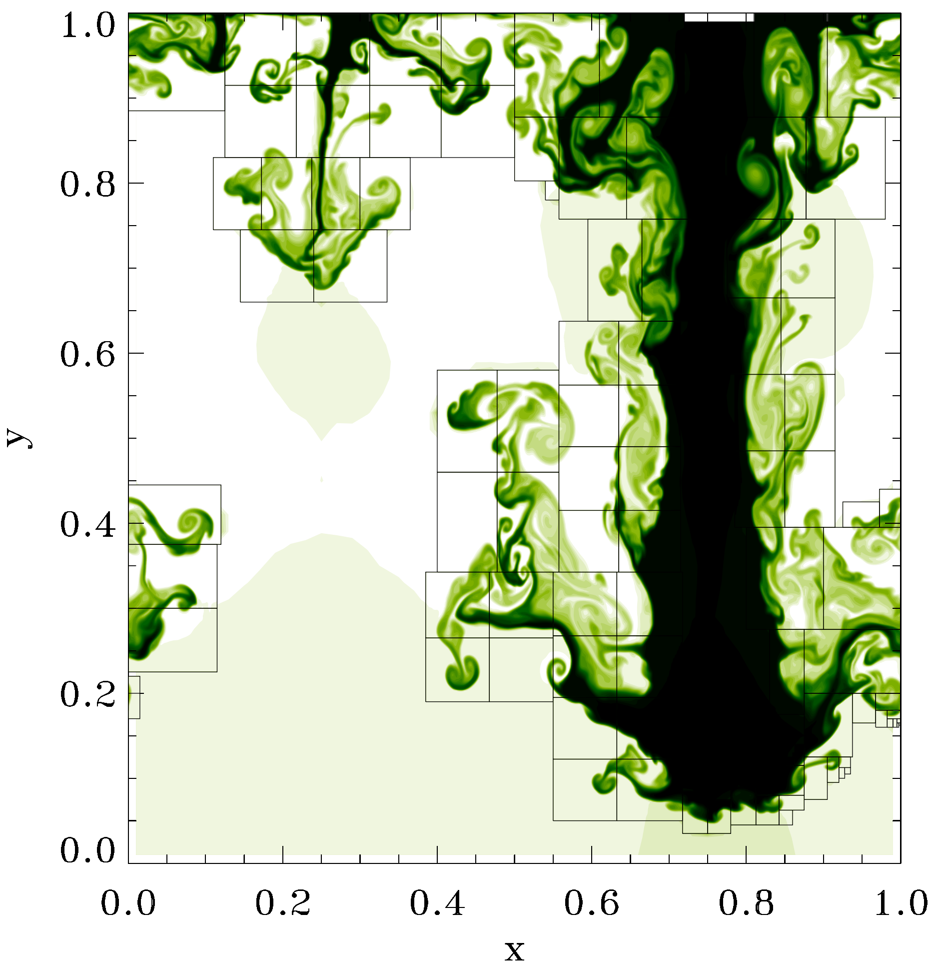
- Since $\mathbf{B}$ and $\hat{\mathbf{B}}$ not in same direction (*magnetic shear* at plasma–vacuum interface), no $\mathbf{k}_0$ exists for which magnetic energies vanish $\Rightarrow$ minimum stabilization when $\mathbf{k}_0$ on average perpendicular to field lines. Rayleigh–Taylor instability may then lead to *interchange instability*: regions of high plasma pressure and vacuum magnetic field are interchanged.

- For dependence on magnitude of $\mathbf{k}_0$, exploit approximations of hyperbolic tangent:

$$\tanh \kappa \equiv \frac{e^\kappa - e^{-\kappa}}{e^\kappa + e^{-\kappa}} \approx \begin{cases} 1 & (\kappa \gg 1: \text{ short wavelength}) \\ \kappa & (\kappa \ll 1: \text{ long wavelength}) \end{cases} . \qquad (95)$$

  Short wavelengths ($k_0 a \, , k_0 b \gg 1$): magnetic $\gg$ gravitational term, system is stable. Long wavelengths ($k_0 a \ll 1$), and $b/a \sim 1$: competition between three terms ($\sim k_0^2$) so that effective *wall stabilization* may be obtained.

Nonlinear evolution from numerical simulation



- Snapshot Rayleigh–Taylor instability for purely 2D hydrodynamic case: density contrast 10, (compressible) evolution.
- Shortest wavelengths grow fastest, 'fingers'/'spikes' develop, shear flow instabilities at edges of falling high density pillars.
- Simulation with AMR-VAC:
  Versatile Advection Code, maintained by Gábor Tóth and Rony Keppens;
  Adaptive Mesh Refinement resolves the small scales.
  www.rijnh.nl/n3/n2/f1234.htm

- **Full nonlinear evolution** (rthd.qt)