Transonic MHD Flows and Shocks

Overview

- **Introduction:** stellar outflows, MHD group diagram and characteristics, connection with the spectral point of view;
- **Discontinuities:** derivation of the MHD jump conditions, application to gas dynamics, four different kinds of discontinuities and shocks in MHD;
- **Transonic equilibria:** symmetric stationary equilibria, self-similar solutions, elliptic and hyperbolic flow regimes, shock conditons;
- **Transonic instabilities:** transonic enigma, Trans-Slow Alfvén Continuum instability, implications for accretion flows;
- **Perspective:** laboratory and astrophysical plasmas from one point of view.

Recall MHD8-37/38:

Solar wind, Parker model

- Coronal plasma at 10^6 K, density drops for increasing r.
  - Pressure gradient drives continuous outflow.
  - Predicted by Parker in 1958, later observed by satellites.
- Model with **hydrodynamic** equations, spherical symmetry:
  - Look for stationary solutions, $\partial/\partial t = 0\,;$
  - Assume isothermal corona (fixed temperature T), include gravity:

$$\frac{d}{dr}(r^2 \rho v) = 0 \qquad \Rightarrow \qquad r^2 \rho v = \mathrm{const}\,,$$

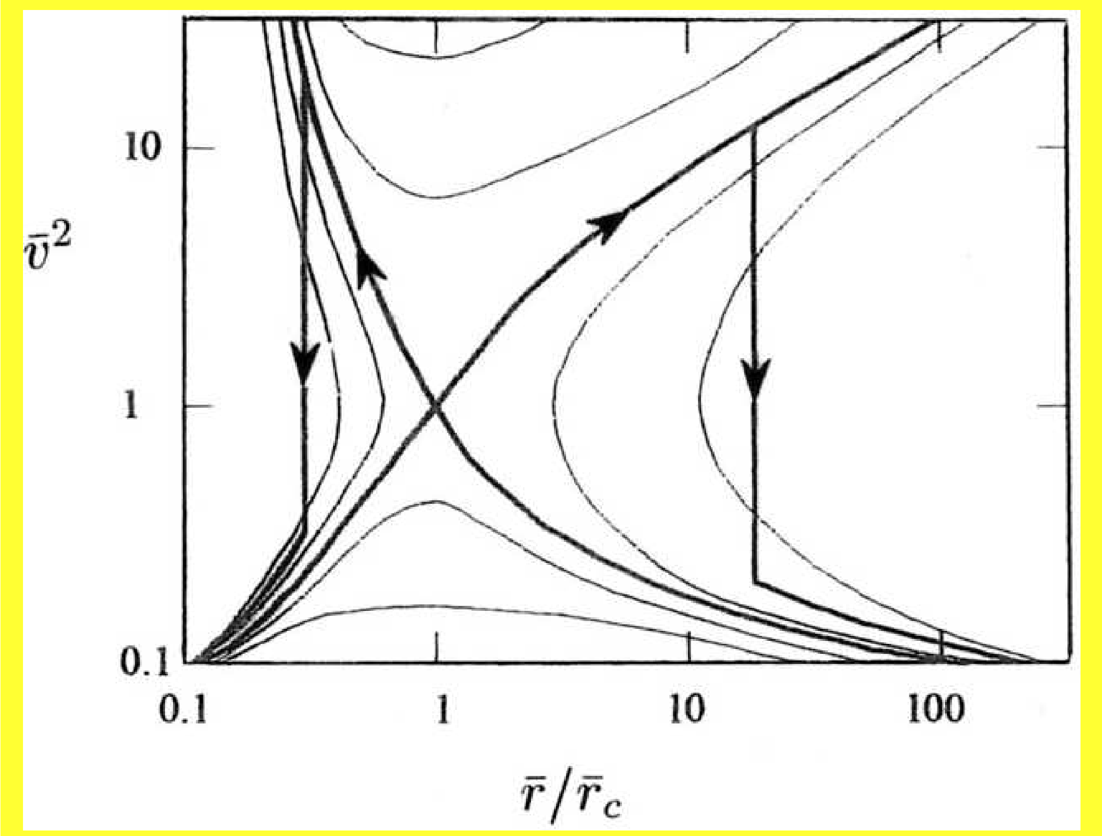
$$\rho v \frac{dv}{dr} + v_{\mathrm{th}}^2 \frac{d\rho}{dr} + G M_\odot \frac{\rho}{r^2} = 0\,;$$

  - Use constant isothermal sound speed $p/\rho \equiv v_{\mathrm{th}}^2$.

- Scale $\bar{v} \equiv v/v_{\rm th}$ (Mach number) and $\bar{r} \equiv r/R_\odot$ to get implicit relation for $\bar{v}(\bar{r})$:

$$F(\bar{v},\bar{r}) \equiv \tfrac{1}{2}\bar{v}^2 - \ln\bar{v} - 2\,\ln\left(\frac{\bar{r}}{\bar{r}_c}\right) - 2\,\frac{\bar{r}_c}{\bar{r}} + \tfrac{3}{2} = C\,, \qquad \bar{r}_c \equiv \tfrac{1}{2}\frac{GM_\odot}{R_\odot v_{\rm th}^2}\,.$$

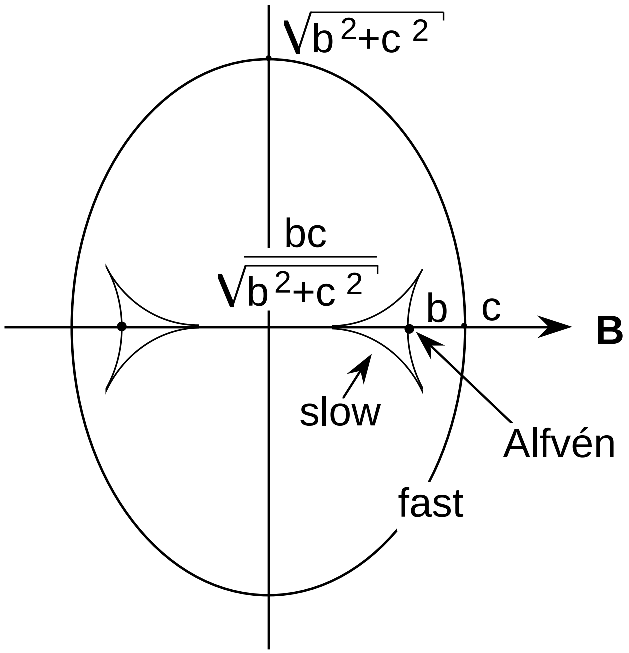
- Two solutions through critical sonic point: solar wind outflow terminating at ISM shock, and (for purpose of illustration) inward accretion also stopped by a shock:

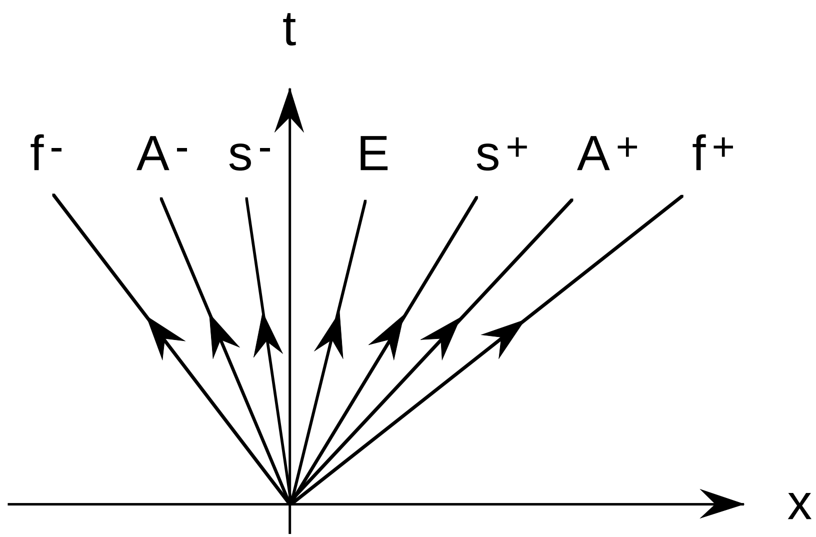


$\Rightarrow$ **Need to investigate MHD counterpart of HD shocks.**

Recall MHD5-29:

MHD group diagram and characteristics





Group diagram is the *ray surface,* i.e. the spatial part of characteristic manifold at certain time t_0.

x-t cross-sections of 7 characteristics (x-axis oblique with respect to $\mathbf{B}$; inclination of entropy mode E indicates plasma background flow).

Connection with spectral point of view

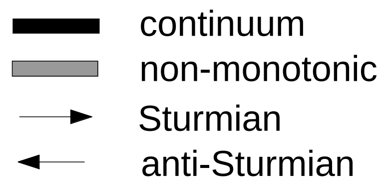
- Short-wavelength limit of spectral structure with *three singular continuous spectra:*

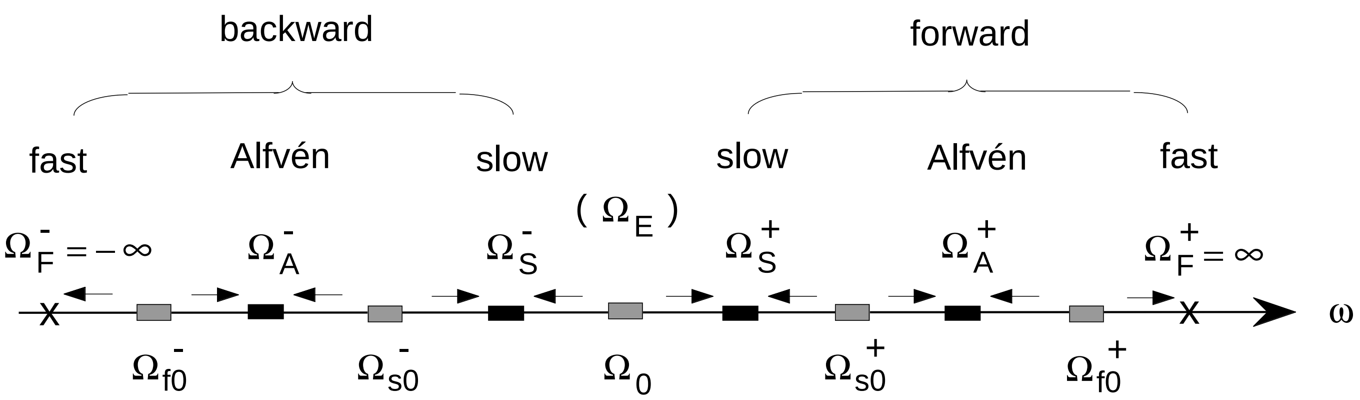
$$\text{slow}: \ \{\omega_S^2(x)\}\,, \quad \text{Alfvén}: \ \{\omega_A^2(x)\}\,, \quad \text{fast}: \ \omega_F^2 \ (=\infty)\,. \tag{1}$$

  For equilibria with flow these continua are *Doppler shifted:*

$$\Omega_S^{\pm} = \pm\omega_S + \mathbf{k}\cdot\mathbf{v}\,, \qquad \Omega_A^{\pm} = \pm\omega_A + \mathbf{k}\cdot\mathbf{v}\,, \qquad \Omega_F^{\pm} = \pm\infty\,. \tag{2}$$

- This yields the following *spectral structure:*



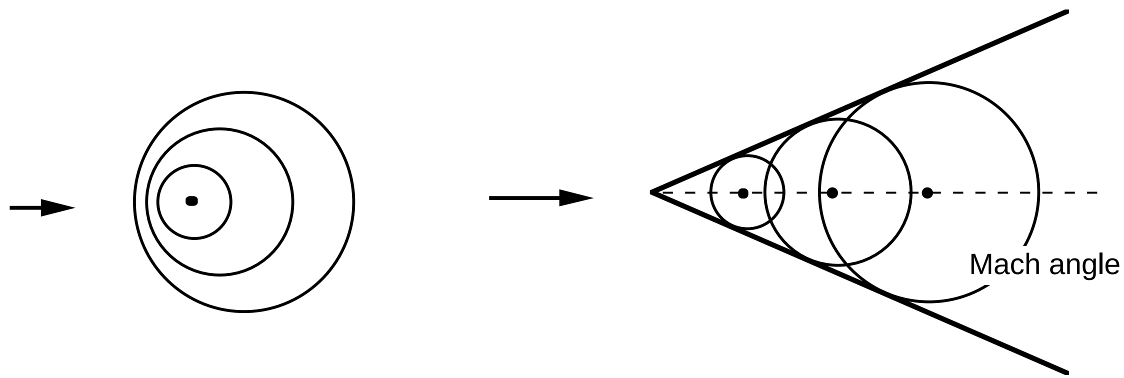


Connection with spectral point of view (cont'd)

- Perturbations of flow propagate along space-time manifolds called *characteristics.* MHD group diagram on S-4 represents snapshot of spatial part of the characteristic. The Lagrangian time derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \qquad (3)$$

  yields temporal phenomena (waves & instabilities) through $\partial/\partial t$, whereas spatial derivative ∇ dominates the description of the stationary equilibrium states. Hence, **linear waves and non-linear stationary equilibria are not separate issues**.



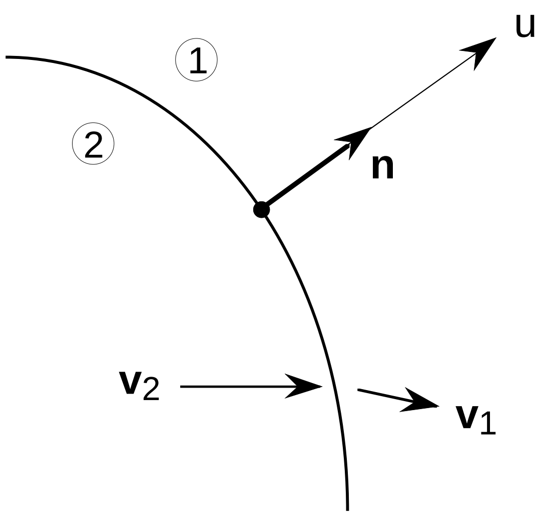
- To get spatial characteristics (or caustics) in MHD, one should construct *tangents to Friedrichs' group diagram* of S-4: much more intricate patterns than in gasdynamics.

Deriving jump conditions

- **Recall MHD4-25/26** with the **general procedure to derive the jump conditions:**
  Integrate conservation equations across shock from ① (undisturbed) to ② (shocked).

- Only contribution from gradient normal to the front:

$$\lim_{\delta\to 0}\int_1^2 \nabla f\, dl = -\lim_{\delta\to 0}\mathbf{n}\int_1^2 \frac{\partial f}{\partial l}\, dl = \mathbf{n}(f_1 - f_2) \equiv \mathbf{n}\,[\![f]\!]\,. \tag{4}$$



- In frame moving with the shock at normal speed u :

$$\left(\frac{\mathrm{D}f}{\mathrm{D}t}\right)_{\rm shock} = \frac{\partial f}{\partial t} - u\frac{\partial f}{\partial l}\ \text{finite}\,,\ \ll \frac{\partial f}{\partial t} \approx u\frac{\partial f}{\partial l} \sim \infty$$

$$\Rightarrow\ \lim_{\delta\to 0}\int_1^2 \frac{\partial f}{\partial t}\, dl = u\lim_{\delta\to 0}\int_1^2 \frac{\partial f}{\partial l}\, dl = -u\,[\![f]\!]\,. \tag{5}$$

- Hence, **jump conditions follow from conservation laws by simply substituting**

$$\nabla f \to \mathbf{n}\,[\![f]\!]\,, \qquad \partial f/\partial t \to -u\,[\![f]\!]\,. \tag{6}$$

Deriving jump conditions (cont'd)

- Conservation of mass,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \Rightarrow \quad -u\,[\![\rho]\!] + \mathbf{n} \cdot [\![\rho \mathbf{v}]\!] = 0\,. \tag{7}$$

- Conservation of momentum,

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot [\,\rho \mathbf{v}\mathbf{v} + (p + \tfrac{1}{2}B^2)\,\mathsf{I} - \mathbf{B}\mathbf{B}\,] = 0$$

$$\Rightarrow \quad -u\,[\![\rho \mathbf{v}]\!] + \mathbf{n} \cdot [\![\rho \mathbf{v}\mathbf{v} + (p + \tfrac{1}{2}B^2)\,\mathsf{I} - \mathbf{B}\mathbf{B}]\!] = 0\,. \tag{8}$$

- Conservation of total energy,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho e + \tfrac{1}{2}B^2) + \nabla \cdot [(\tfrac{1}{2}\rho v^2 + \rho e + p + B^2)\mathbf{v} - \mathbf{v} \cdot \mathbf{B}\mathbf{B}] = 0$$

$$\Rightarrow \quad -u\,[\![\tfrac{1}{2}\rho v^2 + \tfrac{1}{\gamma - 1}p + \tfrac{1}{2}B^2]\!] + \mathbf{n} \cdot [\![(\tfrac{1}{2}\rho v^2 + \tfrac{\gamma}{\gamma - 1}p + B^2)\mathbf{v} - \mathbf{v} \cdot \mathbf{B}\mathbf{B}]\!] = 0\,. \tag{9}$$

- Conservation of magnetic flux,

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0$$

$$\Rightarrow \quad -u\,[\![\mathbf{B}]\!] + \mathbf{n} \cdot [\![\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}]\!] = 0\,, \qquad \mathbf{n} \cdot [\![\mathbf{B}]\!] = 0\,. \tag{10}$$

MHD jump conditions

- **Recall MHD4-27** with the resulting **MHD jump conditions in the shock frame,** obtained from conservation equations *(dropping primes & changing the order!):*

$$[\![\rho v_n]\!] = 0\,, \qquad \textit{(mass)} \qquad (11)$$

$$[\![B_n]\!] = 0\,, \qquad \textit{(normal flux)} \qquad (12)$$

$$\rho v_n\,[\![\mathbf{v}_t]\!] = B_n\,[\![\mathbf{B}_t]\!]\,, \qquad \textit{(tangential momentum)} \qquad (13)$$

$$\rho v_n\,[\![\mathbf{B}_t/\rho]\!] = B_n[\![\mathbf{v}_t]\!]\,, \qquad \textit{(tangential flux)} \qquad (14)$$

$$[\![\rho v_n^2 + p + \tfrac{1}{2}B_t^2]\!] = 0\,, \qquad \textit{(normal momentum)} \qquad (15)$$

$$\rho v_n\,[\![\tfrac{1}{2}(v_n^2 + v_t^2) + (\tfrac{\gamma}{\gamma-1}p + B_t^2)/\rho]\!] = B_n\,[\![\mathbf{v}_t \cdot \mathbf{B}_t]\!]\,. \qquad \textit{(energy)} \qquad (16)$$

$\Rightarrow$ 6 relations for the jumps $[\![v_n]\!]$, $[\![B_n]\!]$, $[\![\mathbf{v}_t]\!]$, $[\![\mathbf{B}_t]\!]$, $[\![p]\!]$, $[\![\rho]\!]$.

- However, drop $\rho v_n\,[\![S]\!] = 0\,,$ replace by $[\![S]\!] \equiv [\![\rho^{-\gamma}p]\!] \leq 0\,.$ *(entropy)* (17)

$\Rightarrow$ 1 constraint on the signs of $[\![\rho]\!]$ and $[\![p]\!]$, such that $S_2 > S_1$:
entropy increases accross shock due to *dissipation* in thin transition layer.

Special case: gas dynamic shocks

- For ordinary gas dynamic shocks ($\mathbf{B} = 0$), the jump conditions reduce to:

$$[\![\rho v_n]\!] = 0\,, \tag{18}$$

$$[\![\rho {v_n}^2 + p]\!] = 0\,, \qquad [\![\mathbf{v}_t]\!] = 0\,, \tag{19}$$

$$[\![\tfrac{1}{2}{v_n}^2 + e + p/\rho]\!] = 0\,, \qquad e = \frac{p}{(\gamma - 1)\rho}\,. \tag{20}$$

Since $[\![\mathbf{v}_t]\!] = 0$, transform to coordinate system moving with tangential flow: $\mathbf{v}_t = 0$. *(This becomes much more intricate in MHD!)* The shock conditions then become:

$$\rho_1 v_1 = \rho_2 v_2\,, \tag{21}$$

$$\rho_1 v_1^2 + p_1 = \rho_2 v_2^2 + p_2\,, \tag{22}$$

$$\tfrac{1}{2} v_1^2 + e_1 + p_1/\rho_1 = \tfrac{1}{2} v_2^2 + e_2 + p_2/\rho_1\,, \qquad e_{1,2} = \frac{p_{1,2}}{(\gamma - 1)\rho_{1,2}}\,. \tag{23}$$

- Solutions for the ratios of quantities on two sides of the shock:

$$\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = 1 - \frac{2(M_1^2 - 1)}{(\gamma + 1)M_1^2}\,, \qquad \frac{p_2}{p_1} = 1 + \frac{2\gamma(M_1^2 - 1)}{\gamma + 1}\,, \qquad M_1^2 \equiv \frac{\rho_1 v_1^2}{\gamma p_1}\,, \tag{24}$$

where **upstream Mach number** M_1^2 is the controlling parameter.

Gas dynamic shocks (cont'd)

- It appears that solutions are found for every value of M_1^2. However, we still have to implement condition (17) to ensure that the entropy increases across the shock:

$$\frac{S_2}{S_1} \equiv \frac{p_2}{p_1}\left(\frac{\rho_2}{\rho_1}\right)^{-\gamma} = \left[1 + \frac{2\gamma(M_1^2-1)}{\gamma+1}\right]\left[1 - \frac{2(M_1^2-1)}{(\gamma+1)M_1^2}\right]^{\gamma} \geq 1\,, \tag{25}$$

This condition can only be satisfied if $M_1^2 \geq 1$, i.e. if **upstream flow is supersonic.** Then, the velocity decreases, whereas the density and the pressure increase across the shock:

$$v_2/v_1 = \rho_1/\rho_2 \leq 1, \qquad p_2/p_1 \geq 1\,, \qquad \text{for} \quad M_1^2 \geq 1\,, \tag{26}$$

whereas

$$M_2^2 \equiv \frac{v_2^2}{v_{s,2}^2} = \frac{\rho_2 v_2^2}{\gamma p_2} = 1 - \frac{(\gamma+1)(M_1^2-1)}{1+\gamma(2M_1^2-1)} \leq 1\,, \tag{27}$$

so that **downstream flow is subsonic.**

MHD discontinuities

- Central to MHD, compared to HD, are the **tangential jump conditions** (13) and (14):

$$(\rho v_n)^2 [\![\mathbf{B}_t/\rho]\!] = \rho v_n B_n [\![\mathbf{v}_t]\!] = B_n^2 [\![\mathbf{B}_t]\!] \,, \tag{28}$$

- They permit to distinguish **four essentially different discontinuities:**

(1) $\rho v_n = 0 \,, B_n \neq 0 \;\Rightarrow\; [\![\mathbf{B}_t]\!] = 0$ $\Rightarrow$ **contact discontinuity;**

(2) $\rho v_n = 0 \,, B_n = 0 \;\Rightarrow$ (28) identity $\Rightarrow$ **tangential discontinuity;**

(3) $\rho v_n^2 = B_n^2 \;\Rightarrow\; [\![\rho]\!] = 0 \,, \; [\![B_t]\!] = \sqrt{\rho}\, [\![v_t]\!]$ $\Rightarrow$ **Alfvén discontinuity;**

(4) $\rho v_n^2 \neq B_n^2 \;\Rightarrow\; [\![\rho]\!] \neq 0$, all relations needed $\Rightarrow$ **magneto-acoustic shock.**

- The first two kinds of discontinuities have been discussed in MHD4-28/30 in relation to the different *laboratory and astrophysical interface models.*

- The latter kinds of discontinuities are *genuine generalizations of the HD shocks.*

Rotational (or Alfvén) discontinuities

- If $\rho v_n \neq 0$ and $[\![\rho]\!] = 0$, the MHD jump conditions give:

$$[\![v_n]\!] = 0\,, \qquad [\![p]\!] = 0\,, \qquad [\![B_t^2]\!] = 0\,, \qquad [\![B_n]\!] = 0\,, \tag{29}$$

$$v_n = B_n/\sqrt{\rho}\,, \qquad [\![\mathbf{v}_t]\!] = [\![\mathbf{B}_t]\!]/\sqrt{\rho} \neq 0\,, \tag{30}$$

  i.e. all thermodynamic variables (p, ρ, e) are continuous, including the entropy (for that reason these discontinuities are not called shocks), and also the magnitude of B, but **direction of B turns through angle about normal.** Also, **normal velocity and jump of tangential velocity are equal to their respective Alfvén velocities.** These are called **rotational, or Alfvén, discontinuities.**

- As always, these dynamical phenomena are central to the MHD picture: The Alfvén discontinuities are precisely intermediate between the slow and the fast magnetosonic shocks, discussed below.

Magneto-acoustic shocks

- If $\rho v_n \neq 0$ and $[\![\rho]\!] \neq 0$, the MHD jump conditions give:

$$\rho v_n [\![\mathbf{v}_t]\!] = B_n [\![\mathbf{B}_t]\!] \,, \tag{31}$$

$$B_n^2 \, [\![B_t]\!] = \rho^2 v_n^2 [\![B_t/\rho]\!] \,, \tag{32}$$

$$[\![\rho v_n^2 + p + \tfrac{1}{2} B_t^2]\!] = 0 \,, \tag{33}$$

$$[\![e]\!] + \left\{ \tfrac{1}{2}(p_1 + p_2) + \tfrac{1}{4}(B_{t1} - B_{t2})^2 \right\} [\![1/\rho]\!] = 0 \,, \tag{34}$$

  i.e. four shock conditions which provide a complete system to determine the jumps across the discontinuity. **Vectors $\mathbf{B}_{t1}$, $\mathbf{B}_{t2}$, $\mathbf{n}$ and $[\![\mathbf{v}_t]\!]$ all lie in the same plane.** These are called **fast, intermediate, and slow magneto-acoustic shocks**. They are genuine generalisations of the gas dynamic shocks for magnetised plasmas.

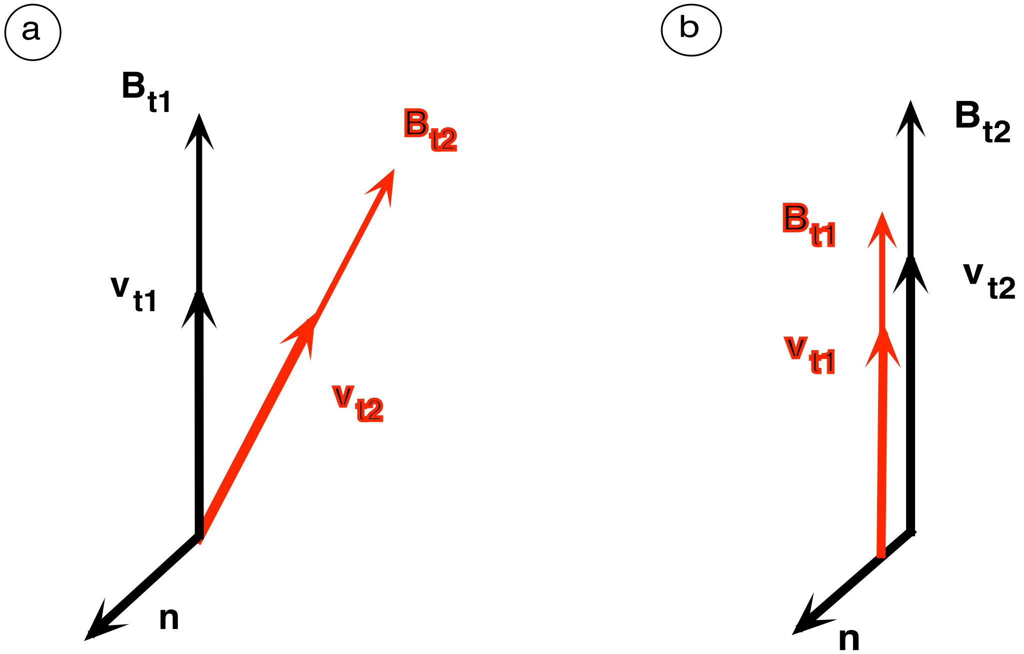
- Note that, because jumps of magnetic field and velocity vectors lie in the same plane, the second relation is not vectorial but just refers to the amplitudes.

- The last condition is obtained from the original energy relation by eliminating the velocity by means of the other jump conditions (quite an exercise!).

Transformation

- The geometric meaning of the jump conditions becomes much clearer when the tangential velocities $\mathbf{v}_{t1}$ and magnetic fields $\mathbf{B}_{t1}$ are aligned by means of a transformation to the **de Hoffman–Teller frame.**

- For **rotational discontinuities** $\mathbf{v}_t$ *and* $\mathbf{B}_t$ *then just rotate over the same angle,* for **magneto-acoustic shocks** *only the amplitudes of* $\mathbf{v}_t$ *and* $\mathbf{B}_t$ *change:*



- For the latter, *switch-on* and *switch-off shocks* may occur (e.g. $\mathbf{B}_{t1} = 0$ but $\mathbf{B}_{t2} \neq 0$).

Magneto-acoustic shock conditions

- As in gas dynamics, relations between upstream and downstream variables are obtained by systematic reduction of jump conditions (31)–(34). First, define **Alfvén Mach numbers** (without subscript A since that is needed for a different purpose):

$$M_1 \equiv \frac{1}{\sqrt{\rho_1}} \frac{\rho v_n}{B_n} , \qquad M_2 \equiv \frac{1}{\sqrt{\rho_2}} \frac{\rho v_n}{B_n} \quad \Rightarrow \quad \frac{M_1^2}{M_2^2} = \frac{\rho_2}{\rho_1} , \tag{35}$$

and its **three threshold values,** determined by p_1 and $B_1^2 \equiv B_{t1}^2 + B_n^2$:

$$M_A^2 \equiv 1 , \qquad M_{s,f}^2 \equiv \frac{\gamma p_1 + B_1^2}{2B_n^2} \left[1 \pm \sqrt{1 - \frac{4\gamma p_1 B_n^2}{(\gamma p_1 + B_1^2)^2}} \right] . \tag{36}$$

- By considerable algebra, the shock conditions can then be reduced to:

$$\begin{aligned} &\left[(\gamma+1)M_2^2 - (\gamma-1)M_1^2 - 2M_s^2 M_f^2\right] (M_2^2 - 1)^2 \\ &= (M_s^2 + M_f^2 - M_s^2 M_f^2 - 1) \left[\gamma M_2^4 - (\gamma-2)M_1^2 M_2^2 - (\gamma+1)M_2^2 + (\gamma-1)M_1^2\right] . \end{aligned} \tag{37}$$

- This condition can be denoted as $f(M_1^2, M_2^2, M_s^2, M_f^2) \geq 0\,.$

  Similarly, the entropy inequality (17) gives a relation $g(M_1^2, M_2^2, M_s^2, M_f^2) \geq 0\,.$

  An example of how to apply such conditions is illustrated on the following pages.

Why interest in MHD shocks?

- For its own sake.
- Because it has important applications in astrophysics.
- Because it is a subject that appears to require a complete reformulation of MHD spectral theory presented thus far:

  $\Rightarrow$ Up till now **split of dynamics in time-independent background equilibrium** (static or stationary) described by **elliptic PDEs** in the spatial domain and **time-dependent perturbations** described by **hyperbolic PDEs** in the space-time domain.

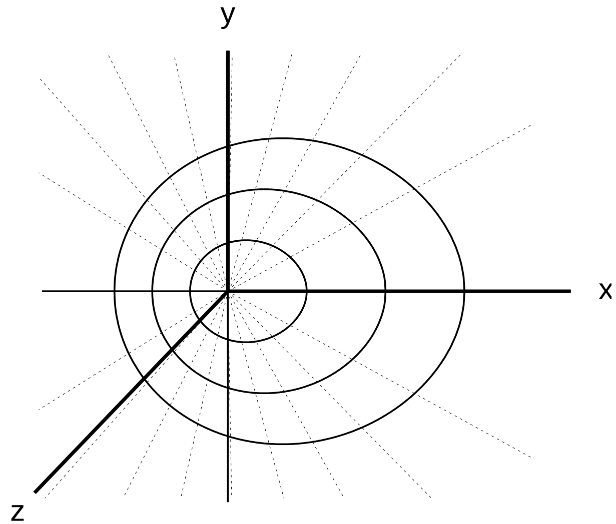
  $\Rightarrow$ With presence of shocks, that split becomes questionable because it may imply that **the equilibrium itself becomes hyperbolic** in the spatial domain.

  (See examples of gas dynamics on page S-6, and 2D transonic MHD flow below.)

  $\Rightarrow$ If we wish to develop **MHD spectroscopy for laboratory and astrophysical plasmas** on an equal footing, the study of **transonic flow and its implication for the MHD waves and instabilities is inescapable.**

Example: Stationary symmetric equilibrium

[Goedbloed & Lifschitz, Phys. Plasmas **4**, 3544 (1997)]



- **Stationary equilibrium** ($\partial/\partial t = 0$) with **translation symmetry** ($\partial/\partial z = 0$).

- Poloidal magnetic field and flow in x–y plane:

$$\mathbf{B}_p = \mathbf{e}_z \times \nabla\psi\,, \quad \text{poloidal flux} \quad \psi(x,y)$$

$$\rho\mathbf{v}_p = \mathbf{e}_z \times \nabla\chi\,, \quad \text{stream function } \chi(\psi)$$

$\Rightarrow$ **poloidal Alfvén Mach number** $M^2(x,y) \equiv \dfrac{\rho v_p^2}{B_p^2} \equiv \dfrac{(\chi')^2}{\rho}\,.$

- Five arbitrary equilibrium flux functions, χ, H (Bernoulli), S (entropy), K (poloidal vorticity/current density), Ω (electric field), collapse onto three: $\Pi_{1,2,3}(\psi)$.

  Core problem: For arbitrary choice of $\Pi_{1,2,3}(\psi)$, determine $\psi(x,y)$ & $M^2(x,y)$.

Variational principle

- Stationary states obtained by minimizing a Lagrangian,

$$\delta \int \mathcal{L}\, dV = 0\,, \qquad \mathcal{L} \equiv \tfrac{1}{2}(1 - M^2)|\nabla\psi|^2 - W(\psi, M^2)\,,$$

where

$$W \equiv \frac{\Pi_1(\psi)}{M^2} - \frac{\Pi_2(\psi)}{\gamma M^{2\gamma}} + \frac{\Pi_3(\psi)}{1 - M^2}\,.$$

$\Rightarrow$ **Nonlinear PDE** for magnetic flux $\psi(x, y)$:

$$\nabla \cdot \left[\, (1 - M^2)\, \nabla\psi \,\right] + \frac{\partial W}{\partial \psi} = 0\,,$$

$\Rightarrow$ **Bernoulli equation** for Mach number $M^2(x, y)$:

$$\tfrac{1}{2}|\nabla\psi|^2 + \frac{\partial W}{\partial M^2} = 0\,.$$

Self-similar solutions

- Assume **master profile** $\pi \equiv \psi^{2-2/\lambda}$,

$$\Pi_1 = \pi(\psi)\,, \quad \Pi_2 = A\,\pi(\psi)\,, \quad \Pi_3 = B\,\pi(\psi)\,,$$

and **self-similarity** in polar coordinates r, θ ,

$$M^{-2} = X(\theta)\,, \qquad \psi = r^{\lambda}\,Y(\theta)\,.$$

$\Rightarrow$ **System of 1st order ODEs for X and Y :**

$$\frac{dX}{d\theta} = \pm \frac{H}{J}\sqrt{2F}$$

$$\frac{dY}{d\theta} = \pm\sqrt{2F}$$

$\Rightarrow$ trajectory $\dfrac{dY}{dX} = \dfrac{J}{H}\,.$

- Special curves in X–Y phase plane:

$F = 0$ – *Bernoulli boundary* (fast & slow flow regimes),

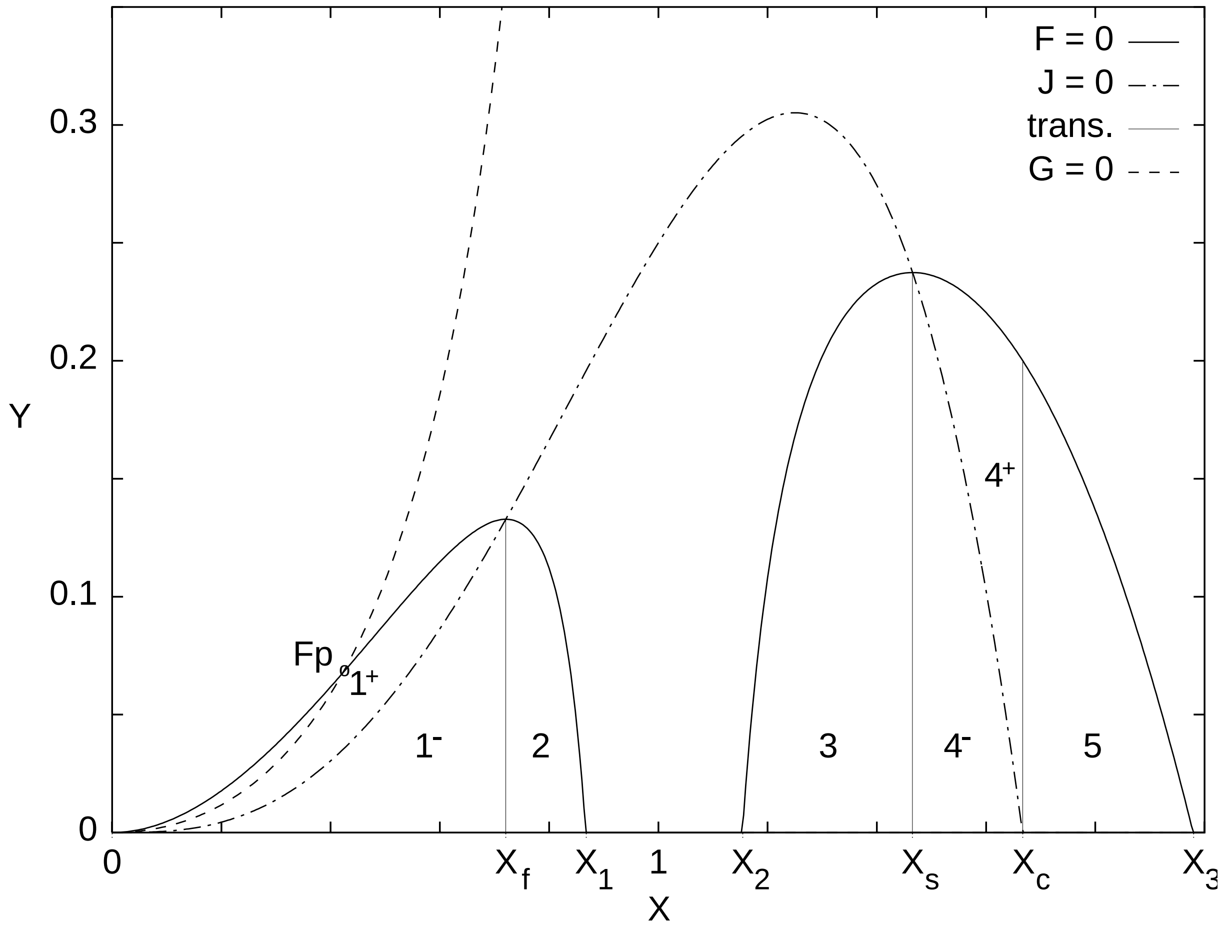
$J = 0$ – *Limiting line characteristic.*

7 flow regimes from Bernoulli equation:

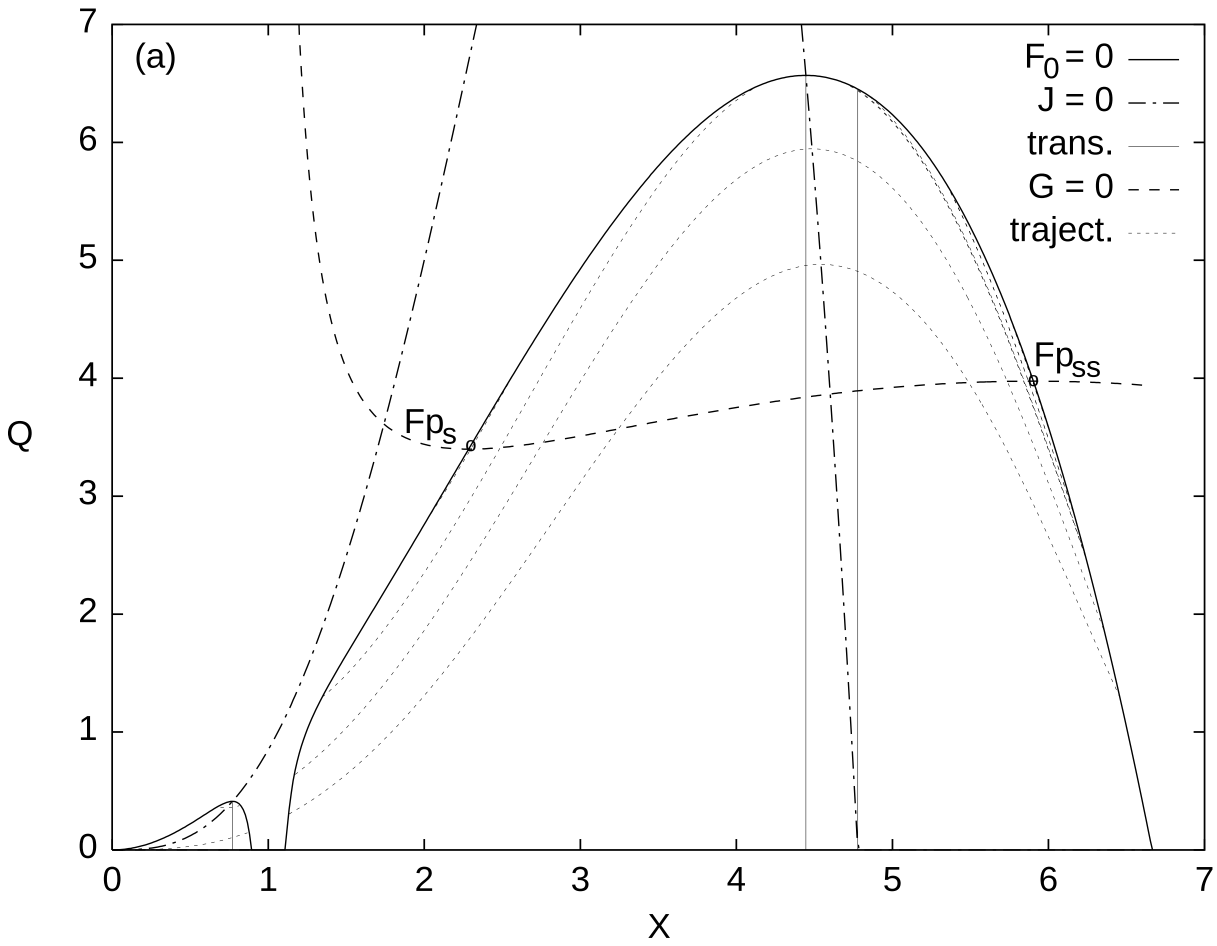
$\mathcal{H}_{ff}\,(1^+)$ (***Fast LL***) $\mathcal{H}_f\,(1^-)$, $\mathcal{E}_f\,(2)$

(*Alfvén gap*)

$\mathcal{E}_s\,(3)$, $\mathcal{H}_s\,(4^-)$ (*Slow LL*) $\mathcal{H}_{ss}\,(4^+)$, $\mathcal{E}_{ss}\,(5)$

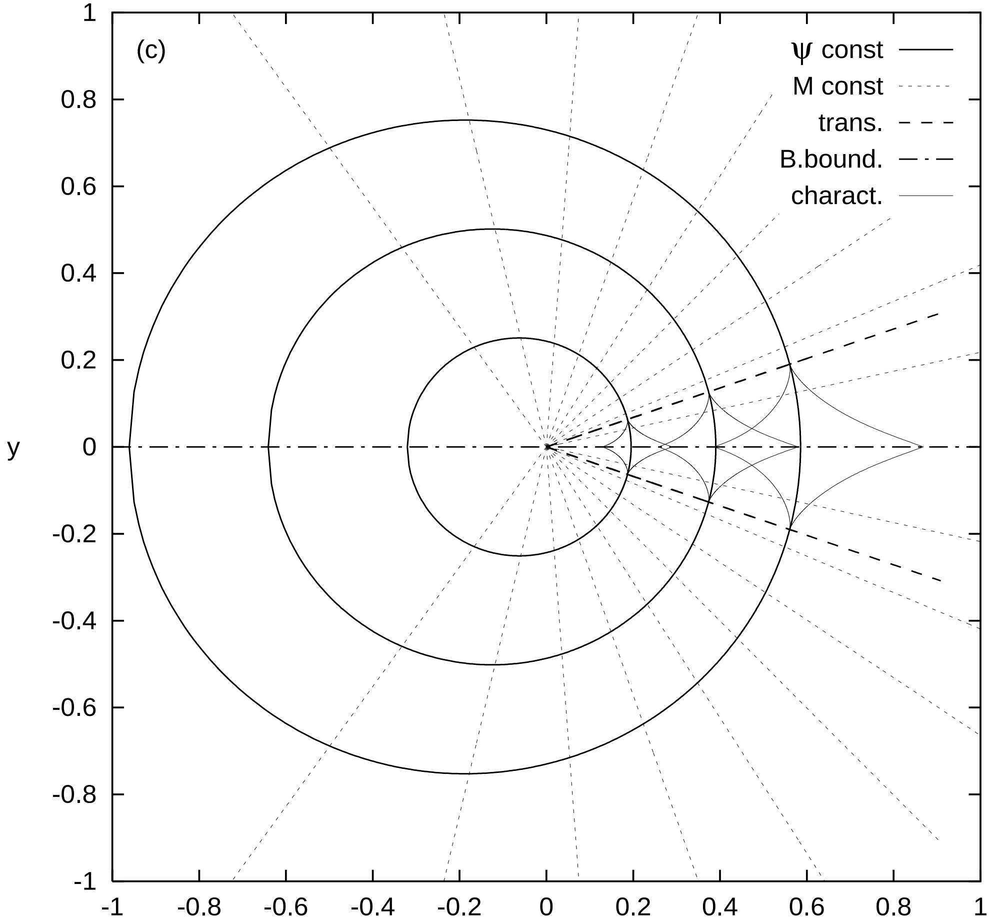


Sub-slow and slow flow trajectories:



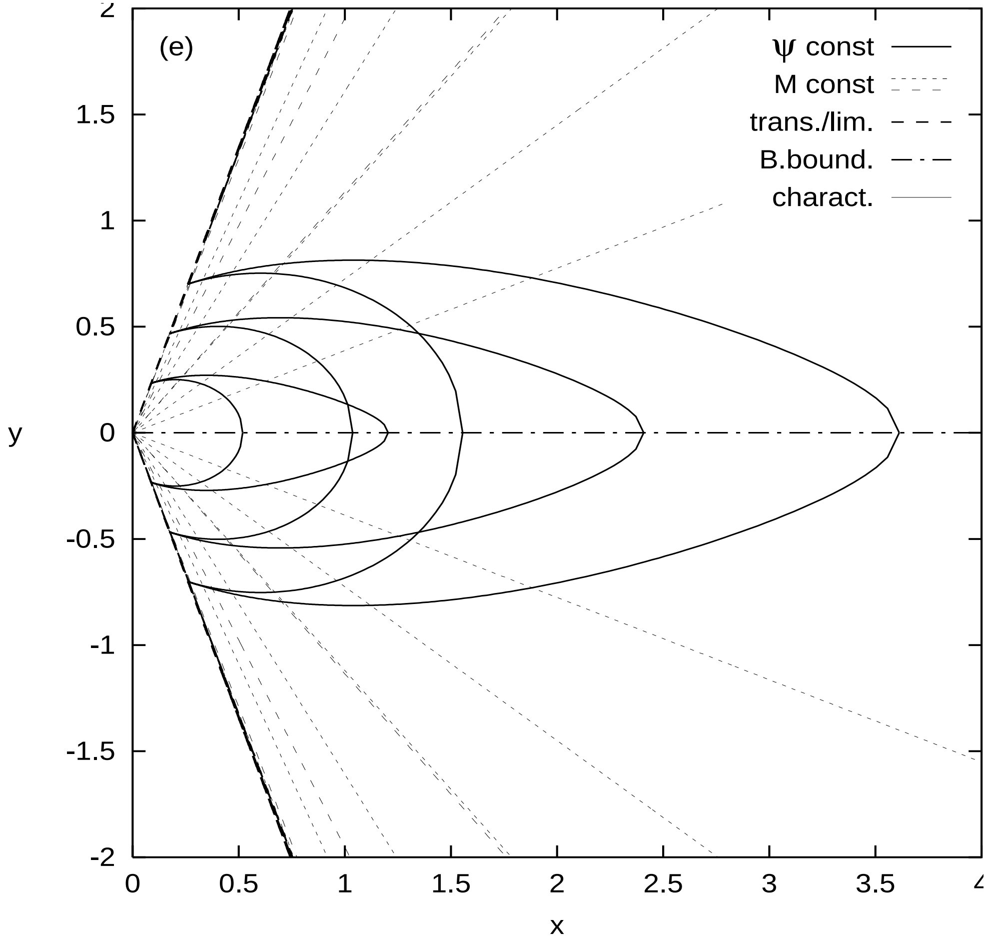
Transition from hyperbolic to elliptic

Sub-slow flow: $\mathcal{H}_{ss}(4^+) \longrightarrow \mathcal{E}_{ss}(5)$:



Flow with a limiting line characteristic

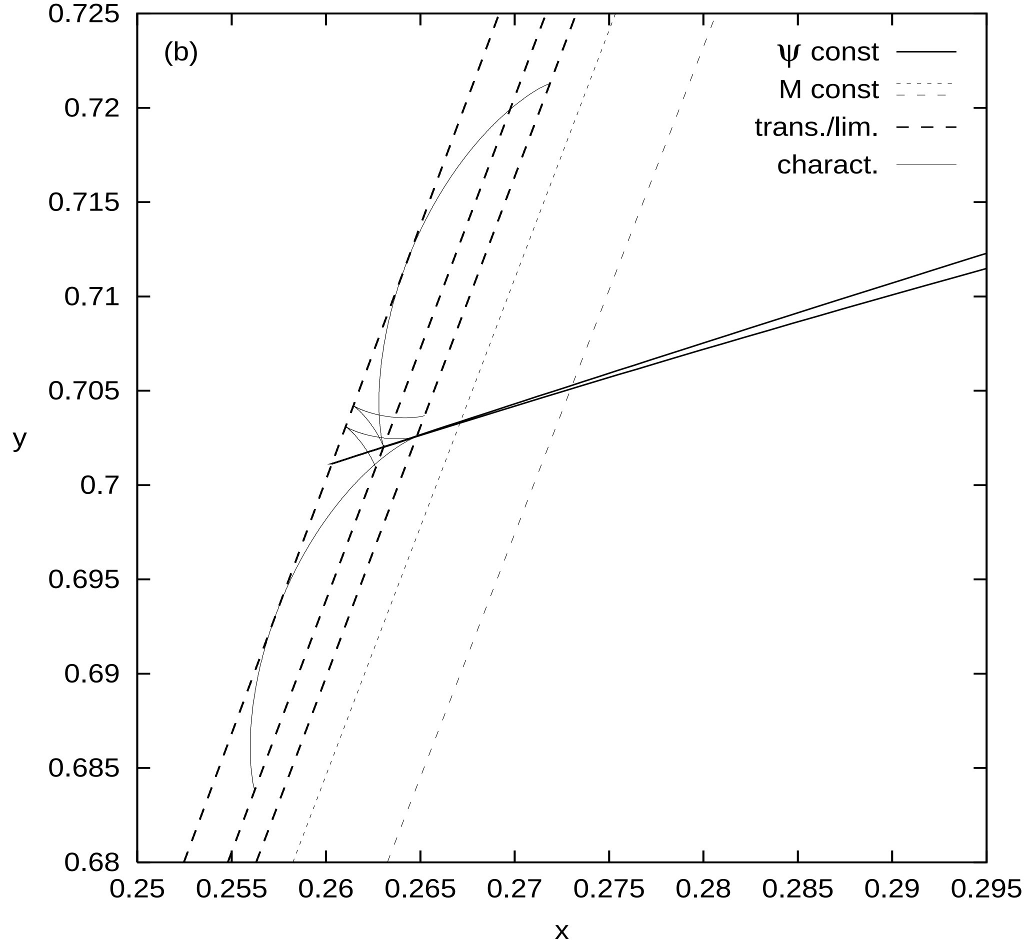
Slow $\mathcal{E}_s\,,\mathcal{H}_s \;\rightarrow\;$ sub-slow $\mathcal{H}_{ss}\,,\mathcal{E}_{ss}$:



Flow pattern 'reflected' by the limiting line?

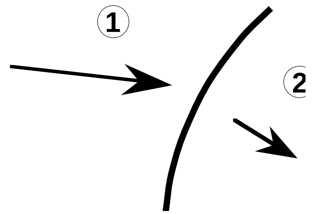
No: Limiting line cannot be crossed!

Flow pattern close to limiting line shows that that streamlines & characteristics are blocked:



⇒ Limiting lines indicate singular discontinuous flow.

Shock conditions



- *Flux* Y and *toroidal field* B continuous: $[\![Y]\!] = 0\,, \quad [\![B]\!] = 0\,.$
- *Inverse Mach number* X and *entropy* A discontinuous:

$$[\![(1-1/X)Y']\!] = 0\,, \quad [\![1/X]\!]\lambda^2 Y^{2/\lambda} + [\![X^2 + (1-1/\gamma - X)AX^\gamma]\!] = 0\,, \quad [\![A]\!] \leq 0\,.$$

- At shock position, 5 parameters:

$$\hat{X}_1 \neq \hat{X}_2\,, \qquad \hat{Y} \equiv \hat{Y}_1 = \hat{Y}_2\,, \qquad \hat{A}_1 \neq \hat{A}_2\,.$$

Eliminating A_2 yields **distilled jump & entropy conditions:**

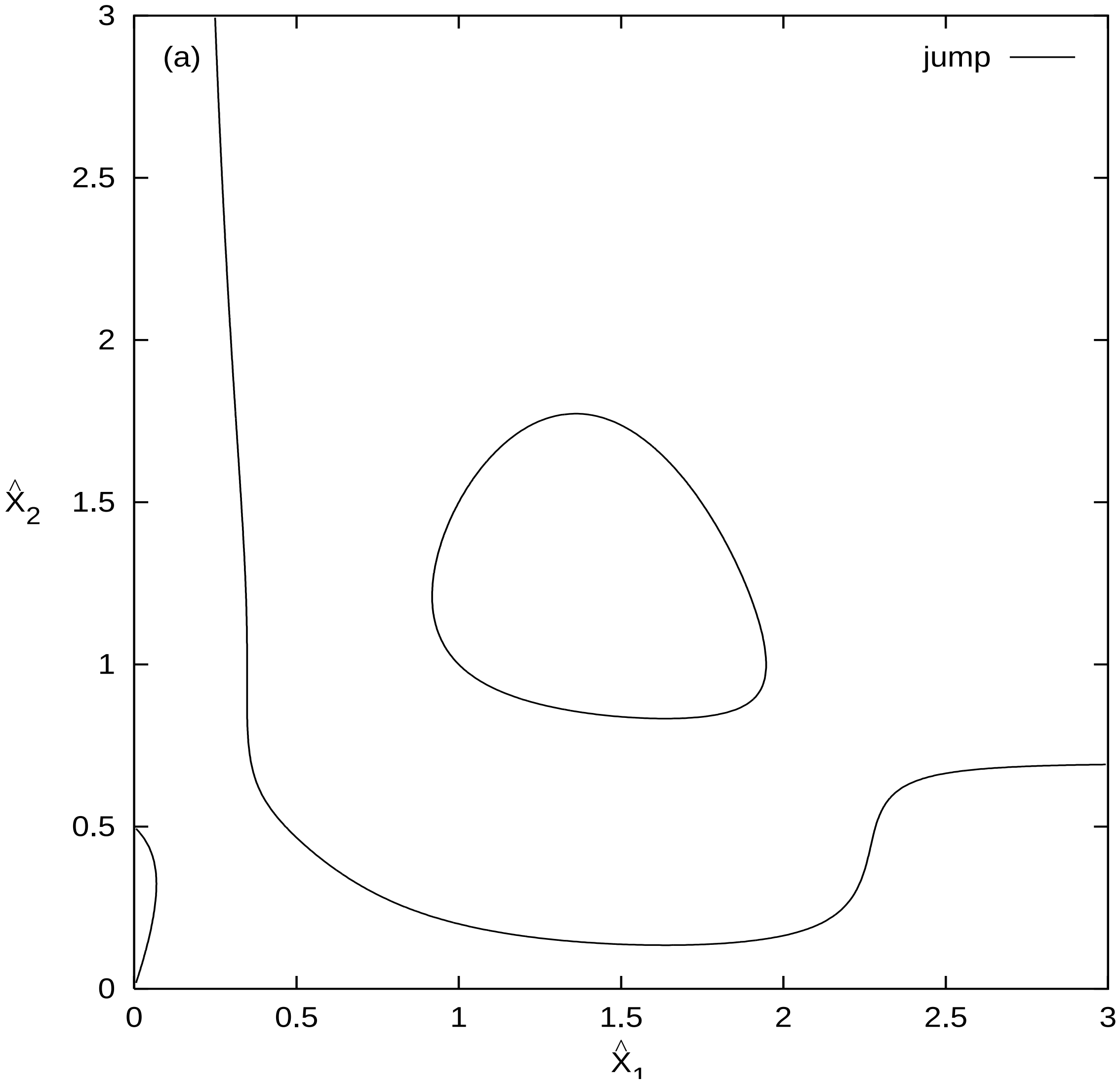
$$f(\hat{X}_1, \hat{X}_2, \hat{Y}, A_1) = 0\,, \qquad g(\hat{X}_1, \hat{X}_2, \hat{Y}, A_1) \geq 0\,,$$

with an additional constraint from the **Bernoulli boundaries:**

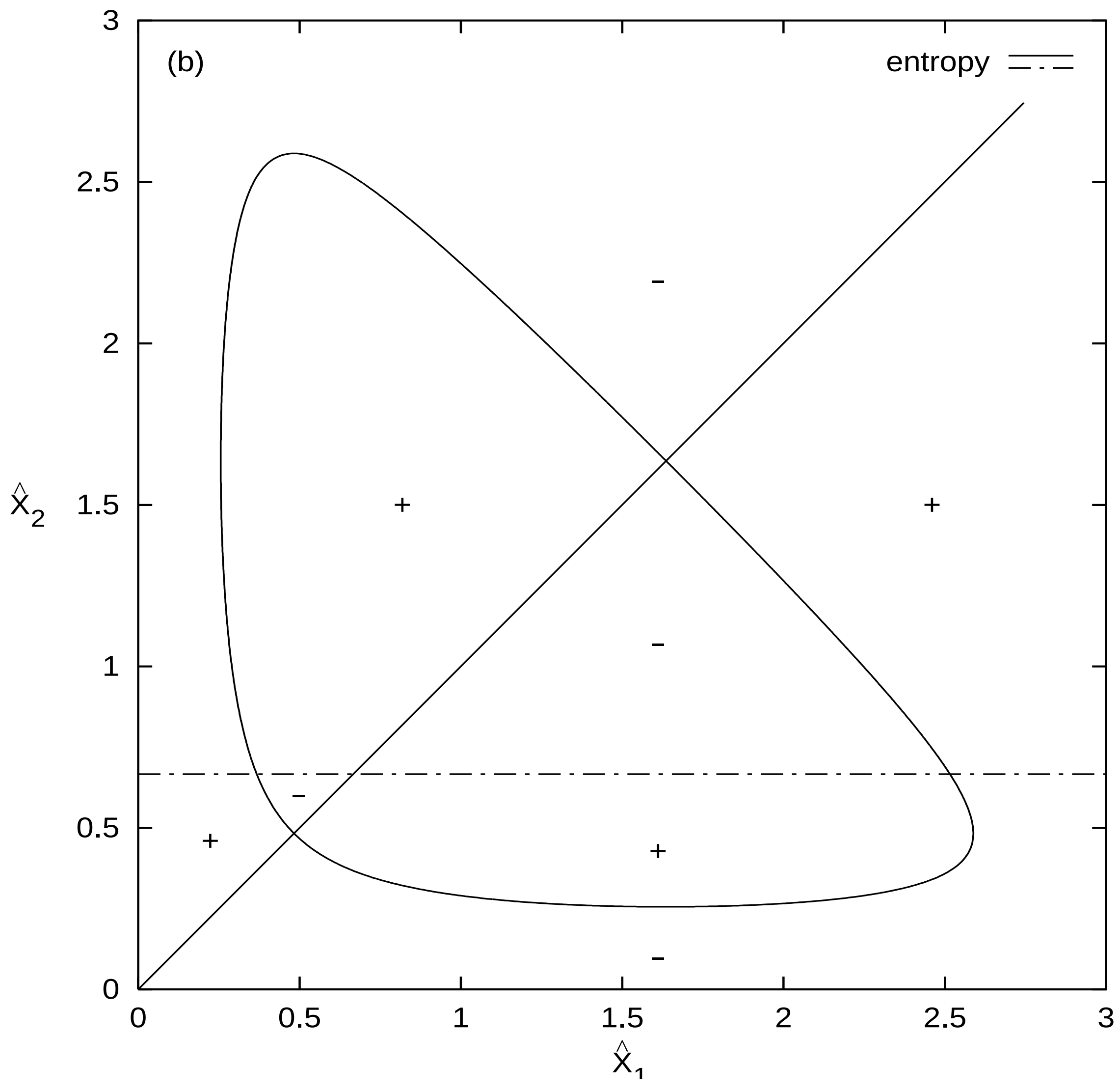
$$F(\hat{X}_1, \hat{Y}, A_1, B) \geq 0\,.$$

- **Procedure:** for given parameters $\hat{Y}$ and A_1, plot f in $\hat{X}_1$—$\hat{X}_2$ plane, and cut out forbidden entropy and Bernoulli parts. This yields the *physically permitted jumps.*

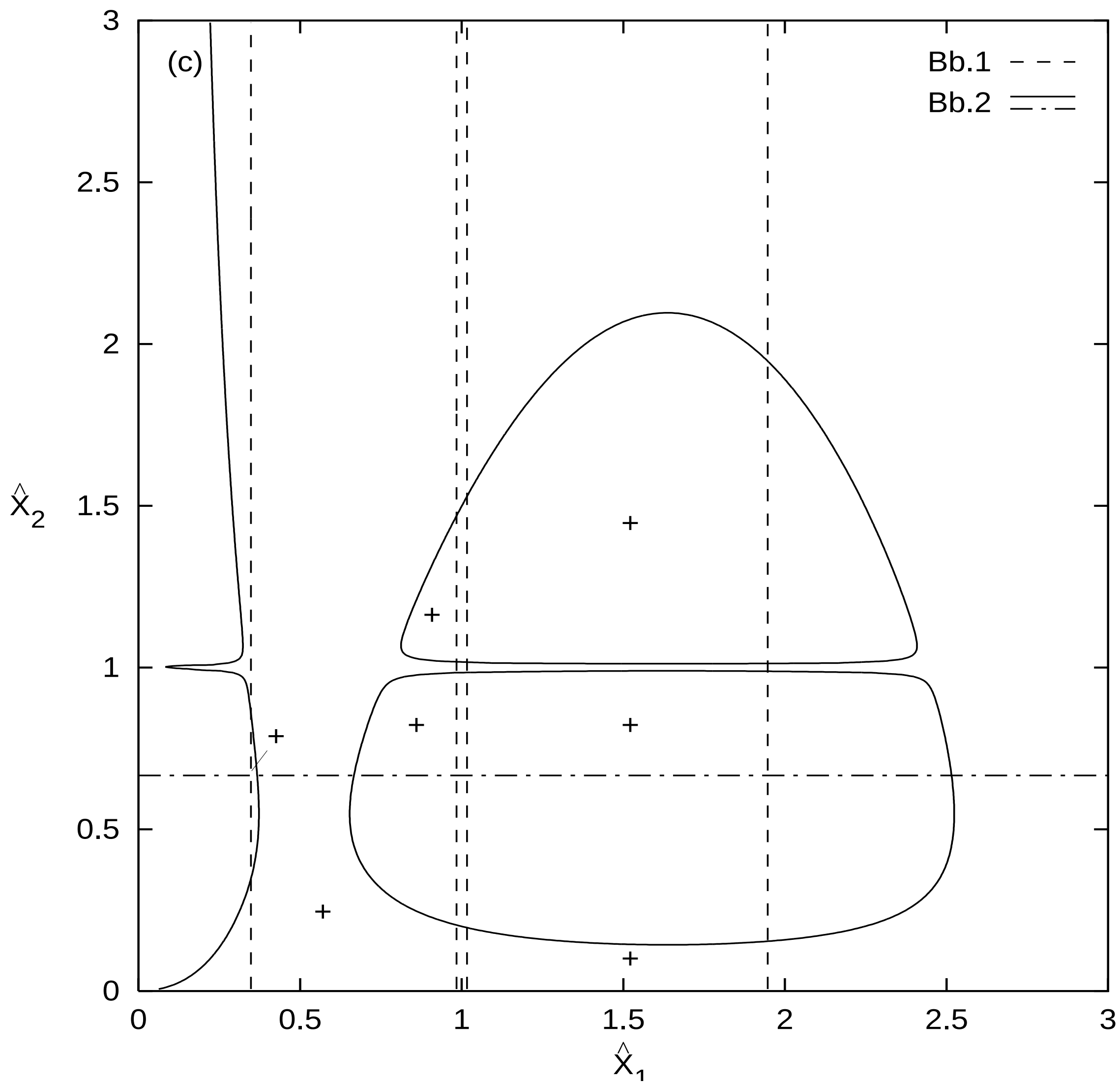
Distilled jump condition



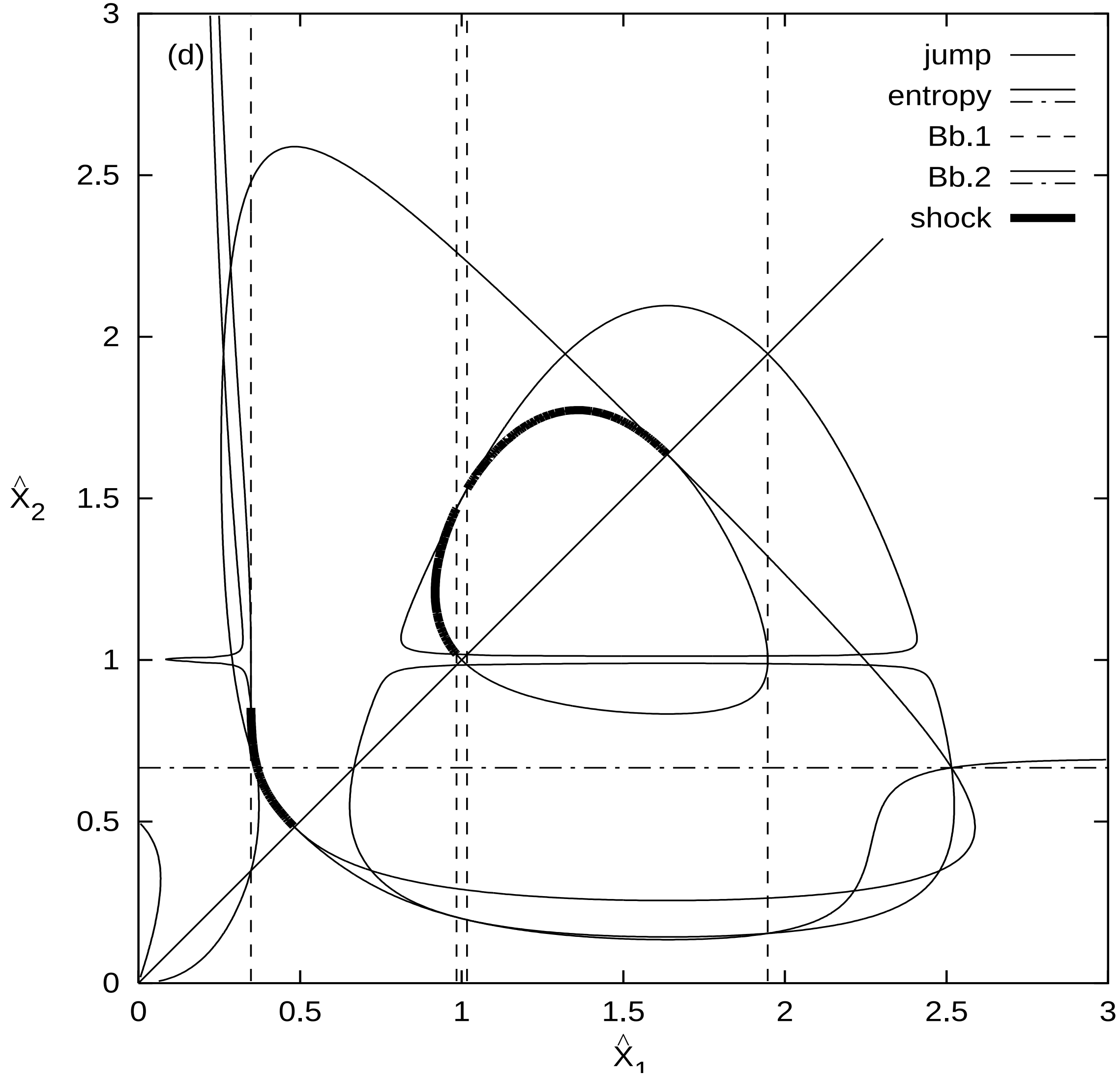
Cutting out forbidden entropy parts



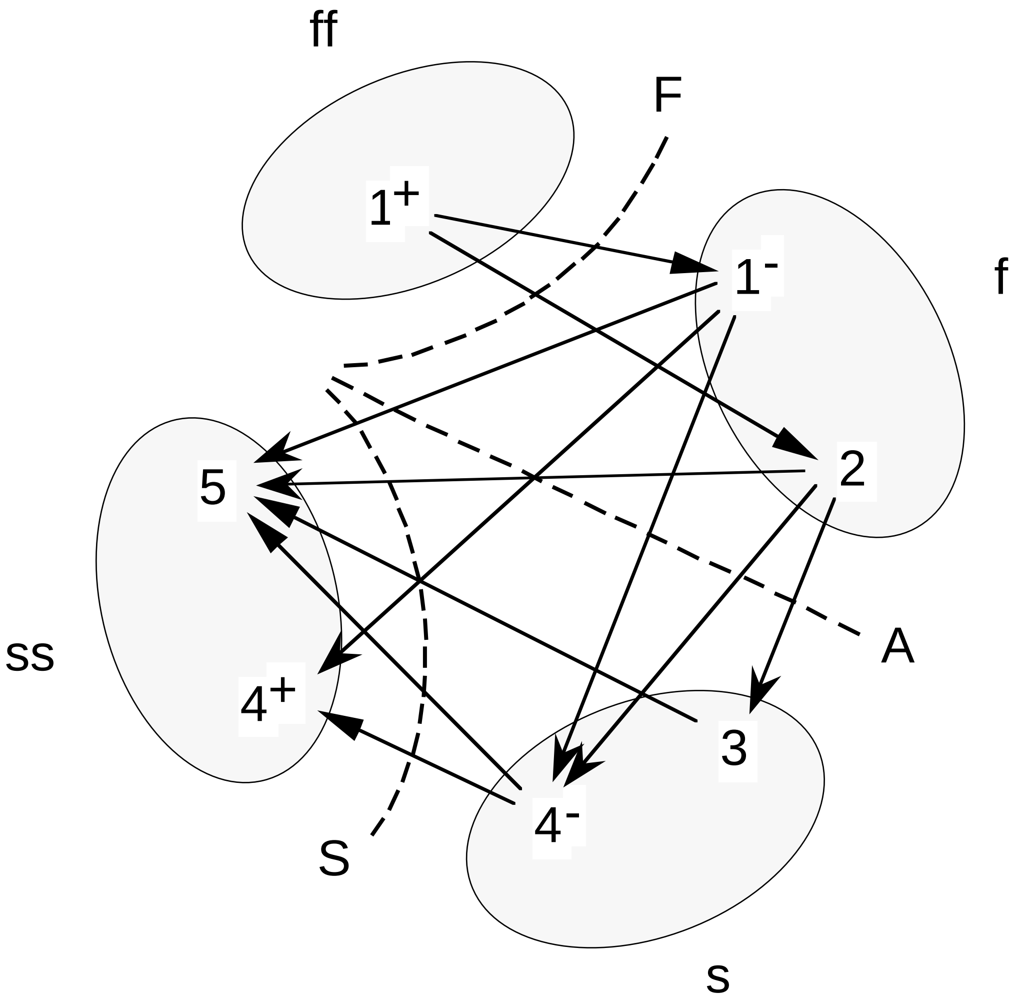
Cutting out forbidden Bernoulli parts



Composite picture: fast, Alfvén & slow shocks



Jumping across the singularities



Connecting the flow regimes: fast, Alfvén & slow shocks.

Summary on transonic equilibria

- There are **four flow regimes**, separated by the limiting lines and the Alfvén gap, and not connected by continuous flows.

- The limiting lines guarantee existence of discontinuous solutions: **Fast shocks jump across the fast limit line, intermediate shocks jump across the Alfvén gap,** and **slow shocks jump across the slow limit line.**

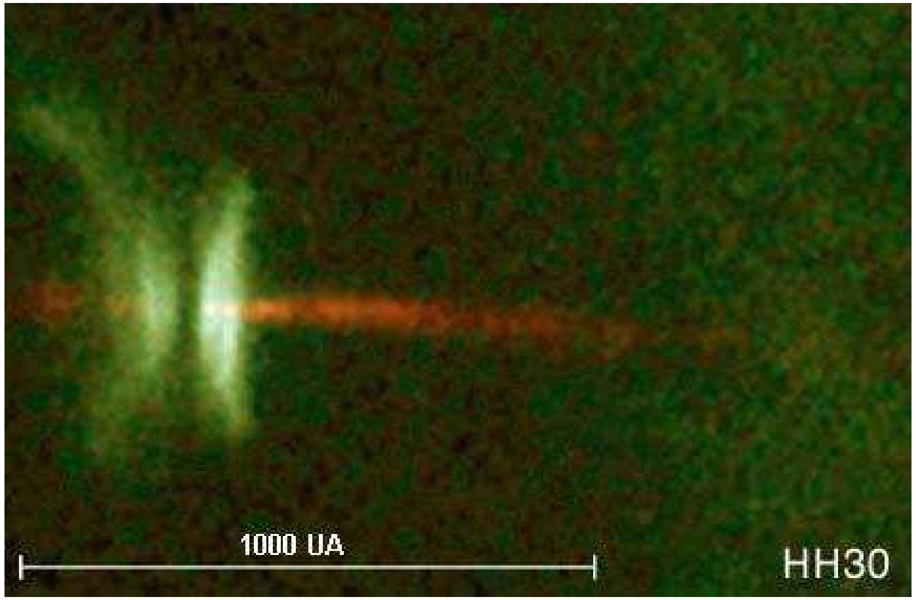
- The three obstacles create the right conditions to produce **three kinds of strongly discontinuous flows** which may be considered as the nonlinear counterparts of the waves of linear MHD.

[Goedbloed & Lifschitz, Phys. Plasmas **4**, 3544 (1997)]

Recall MHDF-20:

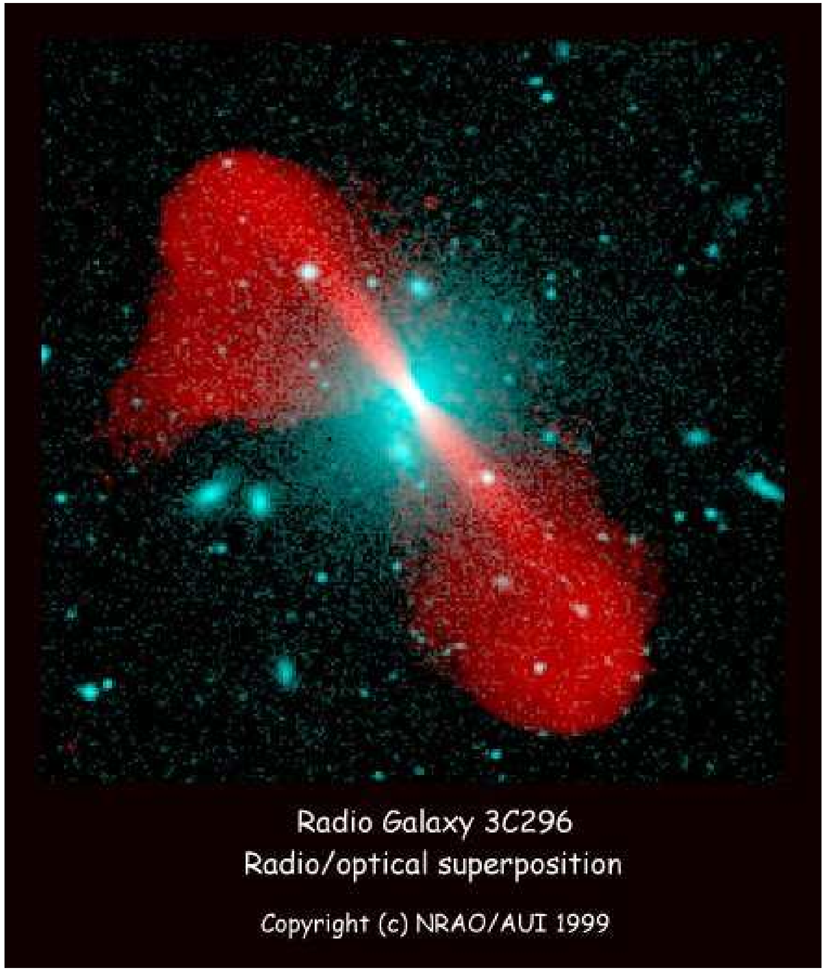
Accretion disk and jets

Young Stellar Object ($M_* \sim 1 M_\odot$)



disk and **jets**

Active Galactic Nucleus ($\sim 10^8 M_\odot$)



disk (optical) and **jets** (radio)

Aim:

Unify laboratory and astrophysical pictures of MHD waves and instabilities

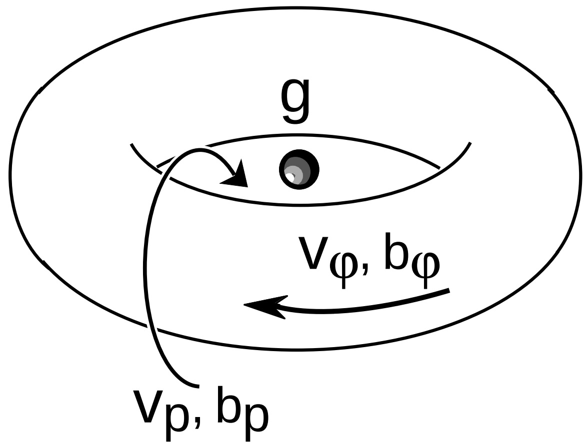
(exploiting scale-independence MHD equations) $\Rightarrow$ **MHD Spectroscopy**

'Historical' development of our own work in this direction:

- MHD spectral theory, with large-scale numerical computations, since 1970s.
- Laboratory plasmas: **MHD spectroscopy for tokamaks.**
  [Goedbloed, Huysmans, Holties, Kerner, Poedts, PPCF **35**, B277 (1993)]
- Astrophysical plasmas: **Magnetoseismology of accretion disks.**
  [Keppens, Casse, Goedbloed, ApJ **579**, L121 (2002)]
- Accretion-ejection needs **anomalous dissipation** $\Rightarrow$ **small-scale instabilities.**
  [Goedbloed, Beliën, van der Holst, Keppens, PoP **11**, 28 (2004)]

$\Rightarrow$ **MHD spectral theory for Transonic Flows (2D)**

Model: 'Superposition' of tokamak and black hole



- **Transonically rotating magnetized (thick) disk about compact object.**
- Accretion speed $\ll$ rotation speeds of the disk
  $\Rightarrow$ **Flow on magnetic surfaces!**
- We investigate:
  Stationary 2D equilibrium $+$ Local instabilities.
- **Gravitational parameter**:

$$\Gamma(\psi) \equiv \frac{\rho G M_*}{R_0 M^2 B^2} \sim \frac{G M_*}{R v_\varphi^2} \quad (= 1 \text{ for Kepplerian flow}).$$

- Analysis and numerics with two new codes,

  FINESSE [Beliën et al. (2002)] & PHOENIX [Blokland, van der Holst et al. (2007)]

  $\Rightarrow$ **Trans-Slow Alfvén Continuum instabilities 'living' on the magnetic surfaces.**

Variational principle for stationary MHD equilibria

- Two unknowns: **pol. flux** ψ, **square pol. Alfvén Mach number** $M^2 \equiv \rho v_p^2 / B_p^2$.
- Equilibrium from minimizing Lagrangian

$$\delta \int \mathcal{L}\, dV = 0\,, \qquad \boxed{\mathcal{L} \equiv \frac{1}{2R^2}(1 - M^2)|\nabla\psi|^2 - \frac{\Pi_1}{M^2} - \frac{\Pi_2}{\gamma M^{2\gamma}} + \frac{\Pi_3}{1 - M^2}}\,,$$

with nonlinear Π_j of **five arbitrary flux functions:**

stream function χ, Bernoulli H, entropy S, electr. pot. Φ, and a function called K.

$\Rightarrow$ 'Grad–Shafranov' equation: $$R^2\, \nabla \cdot \left[R^{-2}(1 - M^2)\, \nabla\psi \right] = \cdots\,, \qquad (38)$$

Bernoulli equation: $$M^2 = M^2(\nabla\psi, \cdots)\,. \qquad (39)$$

- Substituting (13) into (14) gives **transitions from elliptic to hyperbolic flow** when

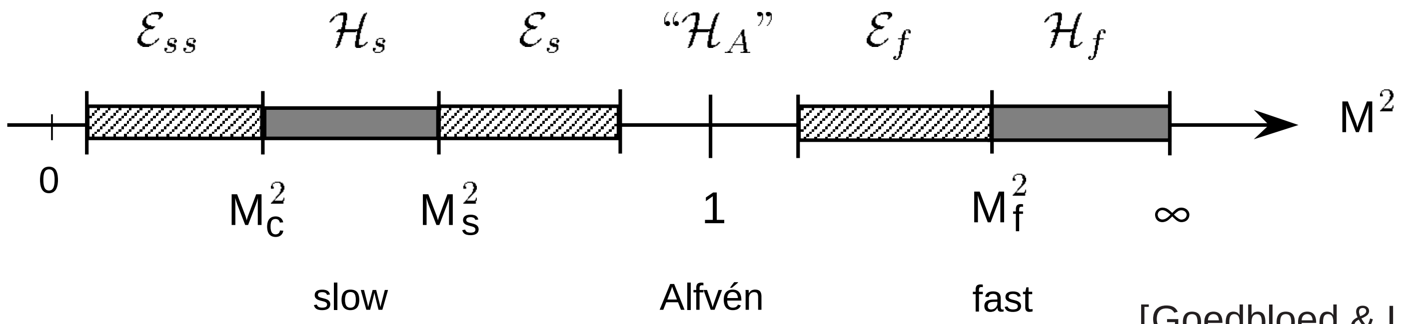
$$\Delta \equiv \frac{\gamma p + B^2}{B_p^2} \frac{M^2 - M_c^2}{(M^2 - M_s^2)(M^2 - M_f^2)} \geq 0\,.$$

$\Rightarrow$ **Slow, (Alfvén,) and Fast hyperbolic regimes,**

Transonic enigma

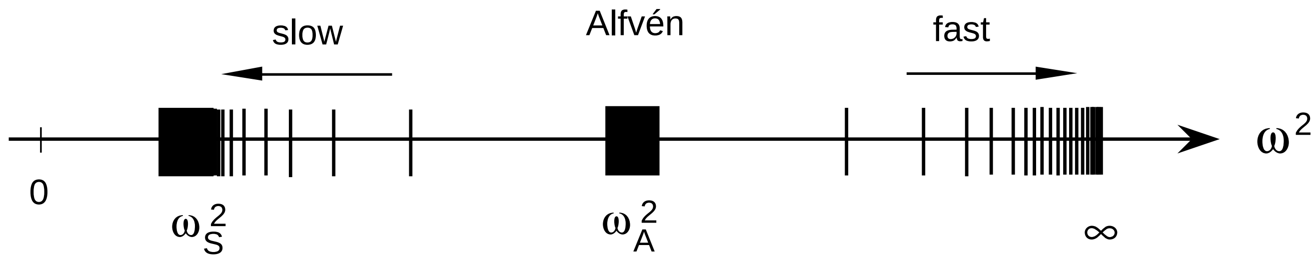
- **Nonlinear stationary states and linear waves no longer independent!**

  – *Hyperbolic flow regimes* delimited by *critical poloidal Alfvén Mach numbers:*



[Goedbloed & Lifschitz (1997)]

  – *Waves* cluster at *continuous spectra* $\{\pm\omega_S\}$, $\{\pm\omega_A\}$, $\pm\infty(\omega_F)$:



- **In hyperbolic regimes, standard tokamak equilibrium solvers diverge!**

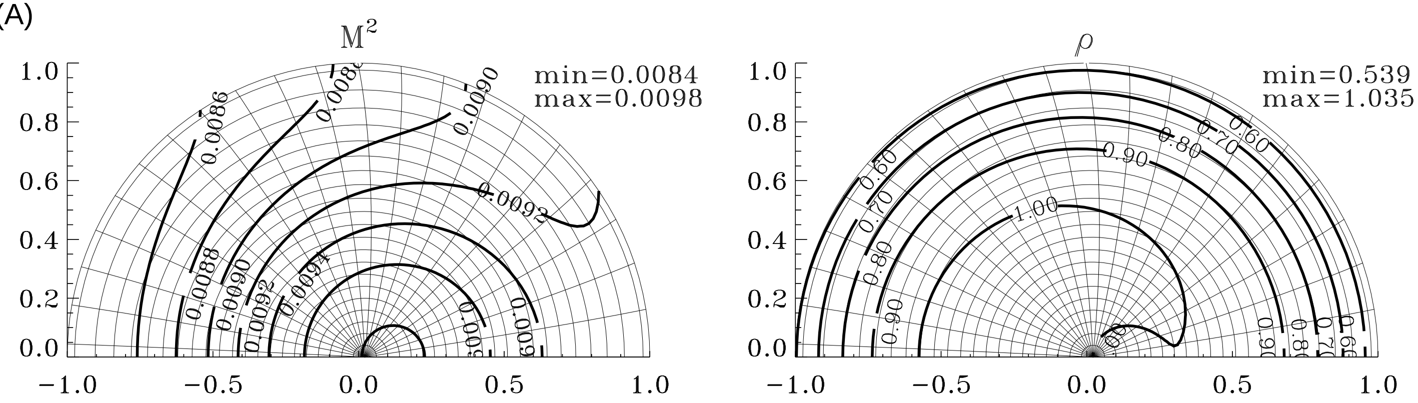
  'Remedy': **calculate in trans-slow elliptic regime,** beyond hyperbolic one.

Trans-slow elliptic equilibria (numerical)

FINESSE code [Beliën et al., JCP **182**, 91 (2002)]

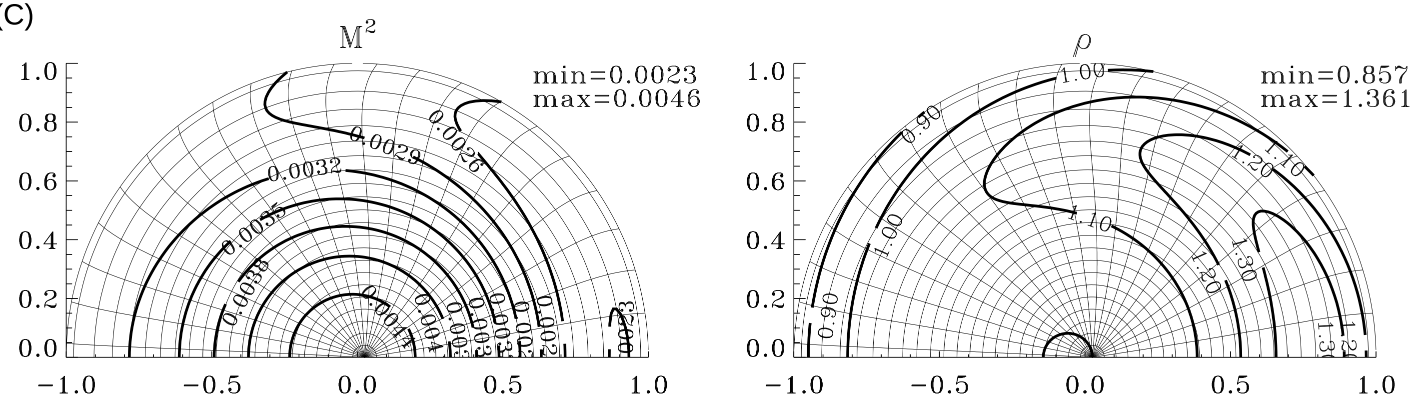
‘Tokamak’ $(\Gamma = 0)$:

(A)



Accretion disk $(\Gamma = 2)$:

(C)



Stability

- **Full MHD spectrum** determined from Frieman-Rotenberg (1960) equation,

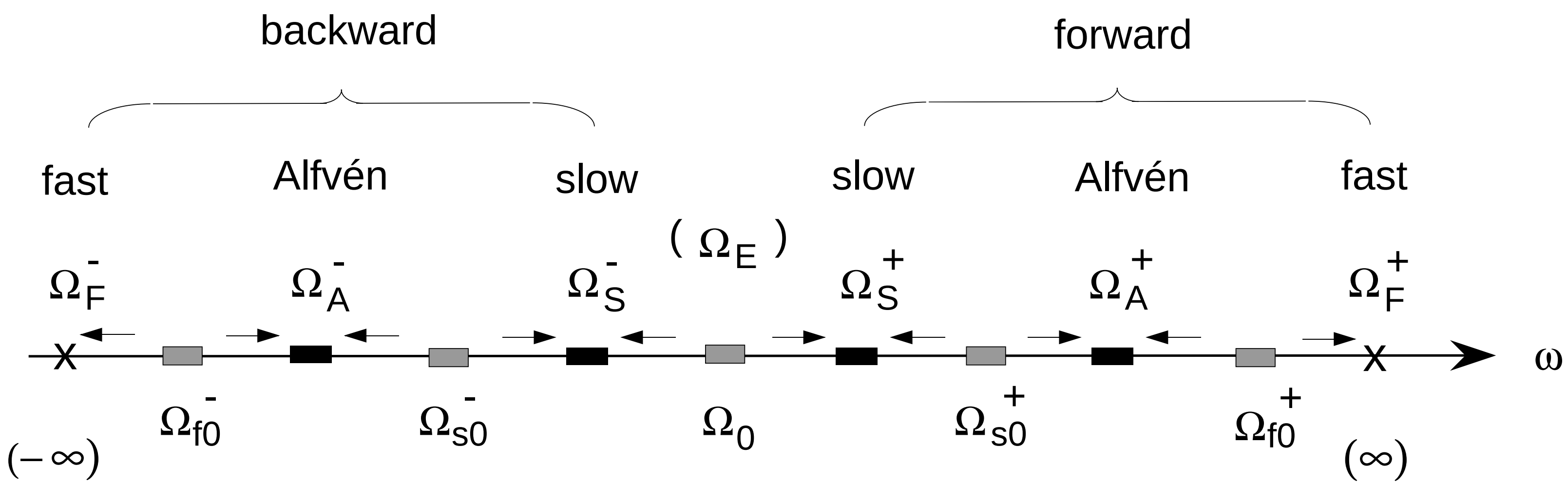
$$\mathbf{F}_{\rm static}(\boldsymbol{\xi}) + \nabla \cdot (\boldsymbol{\xi} \rho \mathbf{v} \cdot \nabla \mathbf{v}) + \rho(\omega + i\mathbf{v} \cdot \nabla)^2 \boldsymbol{\xi} = 0\,,$$

which has complex eigenvalues $\Rightarrow$ **overstable modes.**

- **Six Doppler shifted continuous spectra:**

$$\Omega_S^{\pm} = \pm\omega_S + \Omega_0\,, \quad \Omega_A^{\pm} = \pm\omega_A + \Omega_0\,, \quad \Omega_F^{\pm} = \pm\infty\,, \quad \text{where} \;\; \Omega_0 \equiv \mathbf{k}_0 \cdot \mathbf{v}\,.$$

$\Rightarrow$ Spectral structure for stationary plasmas [Goedbloed et al. PoP **11**, 4332 (2004)]:



- **Torus: Instability by coupling harmonics $e^{im\vartheta}$ of Alfvén and slow continua.**

Transonic continuum modes

- **Singular modes** localized about **single magnetic / flow surface:**

$$\xi_{\perp,\parallel} \approx \delta(\psi - \psi_0)\, \hat{\xi}_{\perp,\parallel}(\vartheta)\, e^{in\varphi} \quad \Rightarrow \text{ EVP } \boxed{\hat{\mathbf{A}} \cdot \hat{\mathbf{V}} = \hat{\mathbf{B}} \cdot \hat{\mathbf{V}}}\,, \quad \hat{\mathbf{V}} \equiv (\hat{\xi}_\perp\,, \hat{\xi}_\parallel)^{\mathrm{T}}\,,$$

$$\hat{\mathbf{A}} \equiv \begin{pmatrix} \mathcal{F}\frac{R^2B_p^2}{B^2}\mathcal{F} - (M^2 - M_c^2)\frac{B^2}{\rho^2}\left[\partial(\frac{\rho R B_\varphi}{B^2})\right]^2 & -i(M^2 - M_c^2)\frac{B^2}{\rho^2}\left[\partial(\frac{\rho R B_\varphi}{B^2})\right]\mathcal{F}\rho \\ i\rho\mathcal{F}(M^2 - M_c^2)\frac{B^2}{\rho^2}\left[\partial(\frac{\rho R B_\varphi}{B^2})\right] & \mathcal{F}M_c^2B^2\mathcal{F} + \rho\left[\partial((M^2 - M_c^2)\frac{B^2}{\rho^2}\,\partial\rho)\right] \end{pmatrix},$$

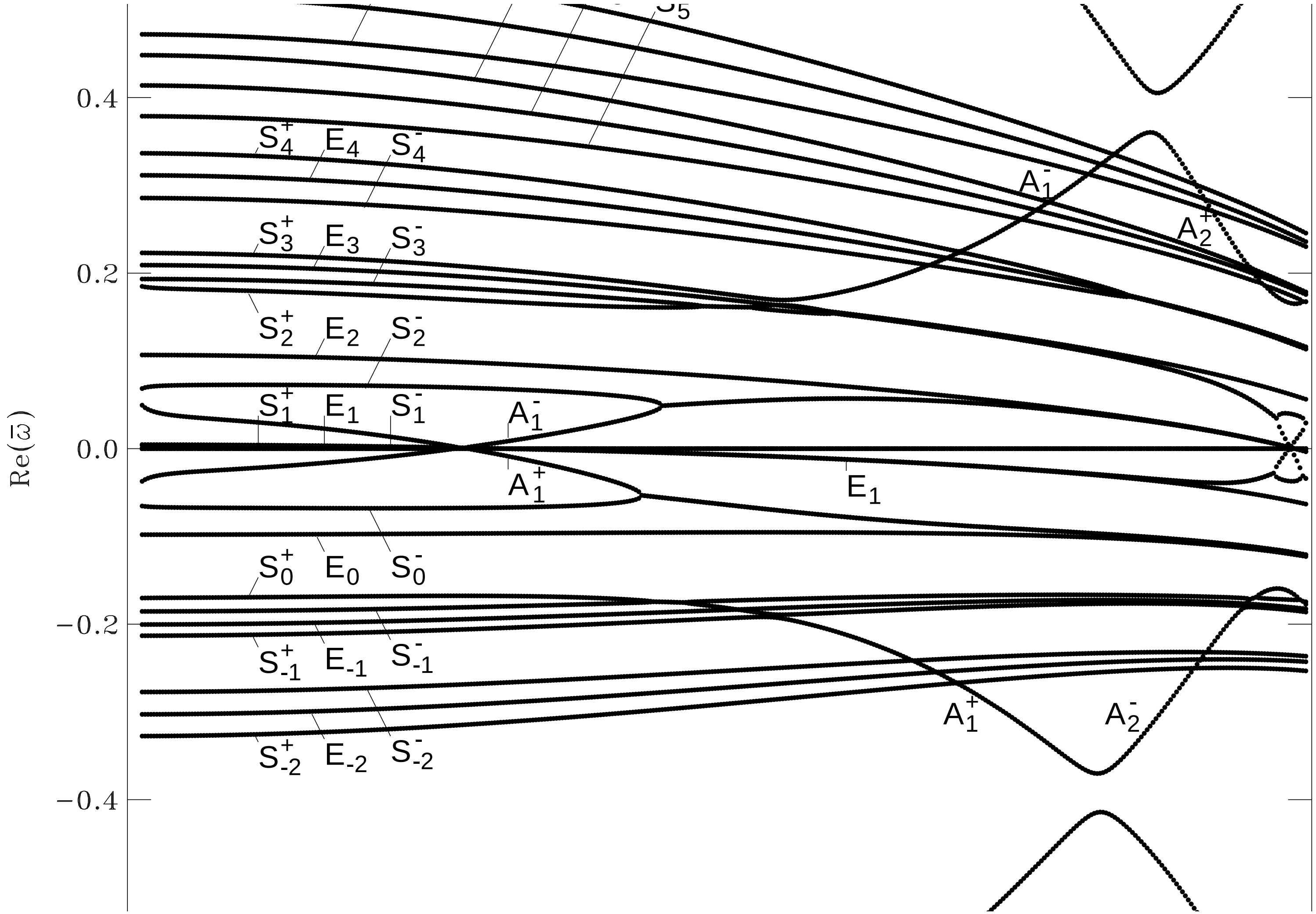
$$\hat{\mathbf{B}} \equiv \begin{pmatrix} (\sqrt{\rho}\,\widetilde{\omega} - \mathcal{F}M)\frac{R^2B_p^2}{B^2}(\sqrt{\rho}\,\widetilde{\omega} - M\mathcal{F}) & -i\alpha\sqrt{\rho}\,\widetilde{\omega} \\ i\alpha\sqrt{\rho}\,\widetilde{\omega} & (\sqrt{\rho}\,\widetilde{\omega} - \mathcal{F}M)B^2(\sqrt{\rho}\,\widetilde{\omega} - M\mathcal{F}) \end{pmatrix}.$$

Doppler shifted frequency $\widetilde{\omega} \equiv \omega - n\Omega$ in frame rotating with Ω (where $\mathbf{E} = 0$).

- **Always unstable in the trans-slow ($M^2 > M_c^2$) flow regimes!**

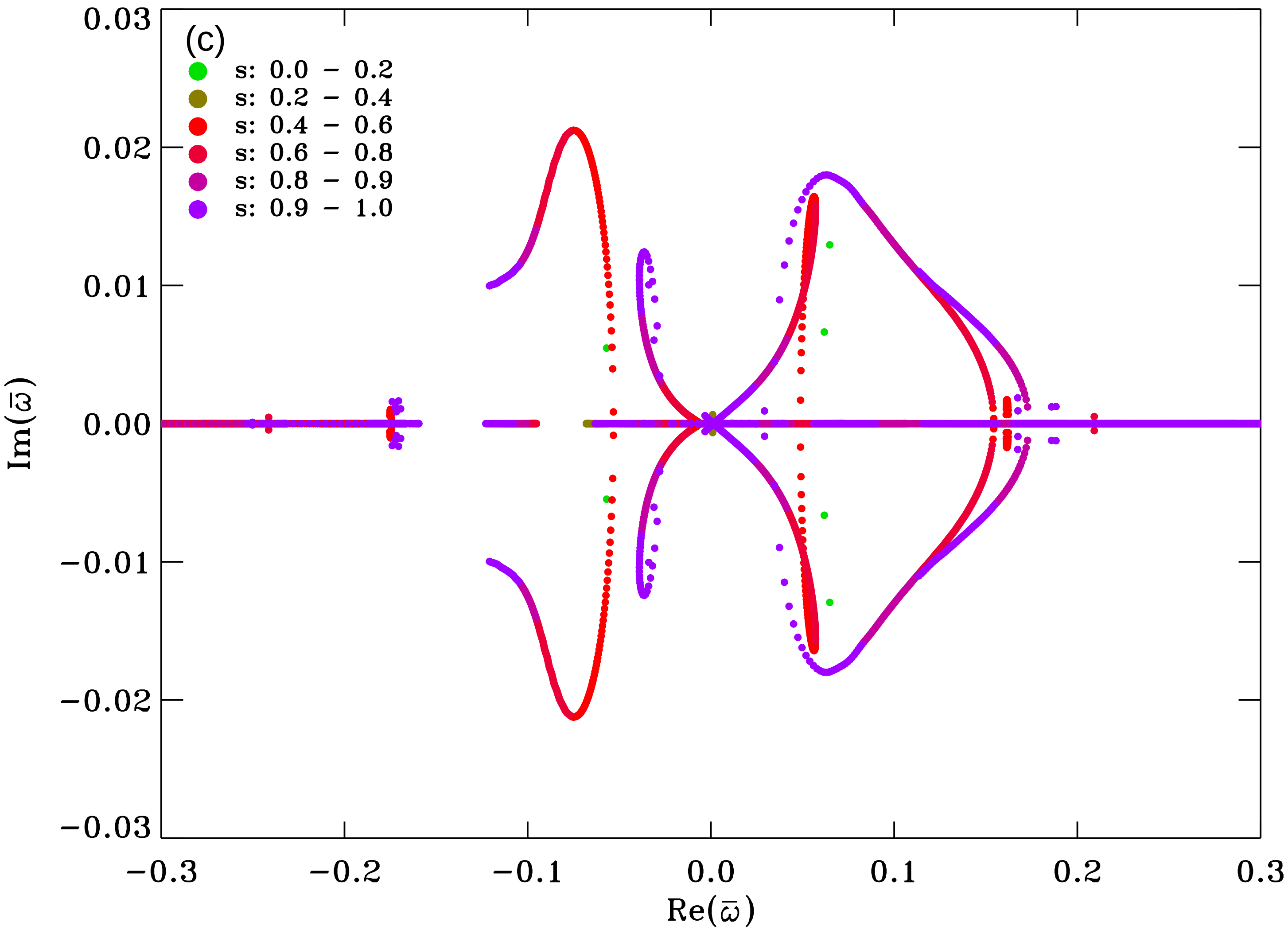
Mode labeling

$A_1^{\pm}$ *branches strongly interact with* $S_0^{\pm}$ *and* $S_2^{\pm}$ *:*



‘Tokamak’ ($\Gamma = 0$)

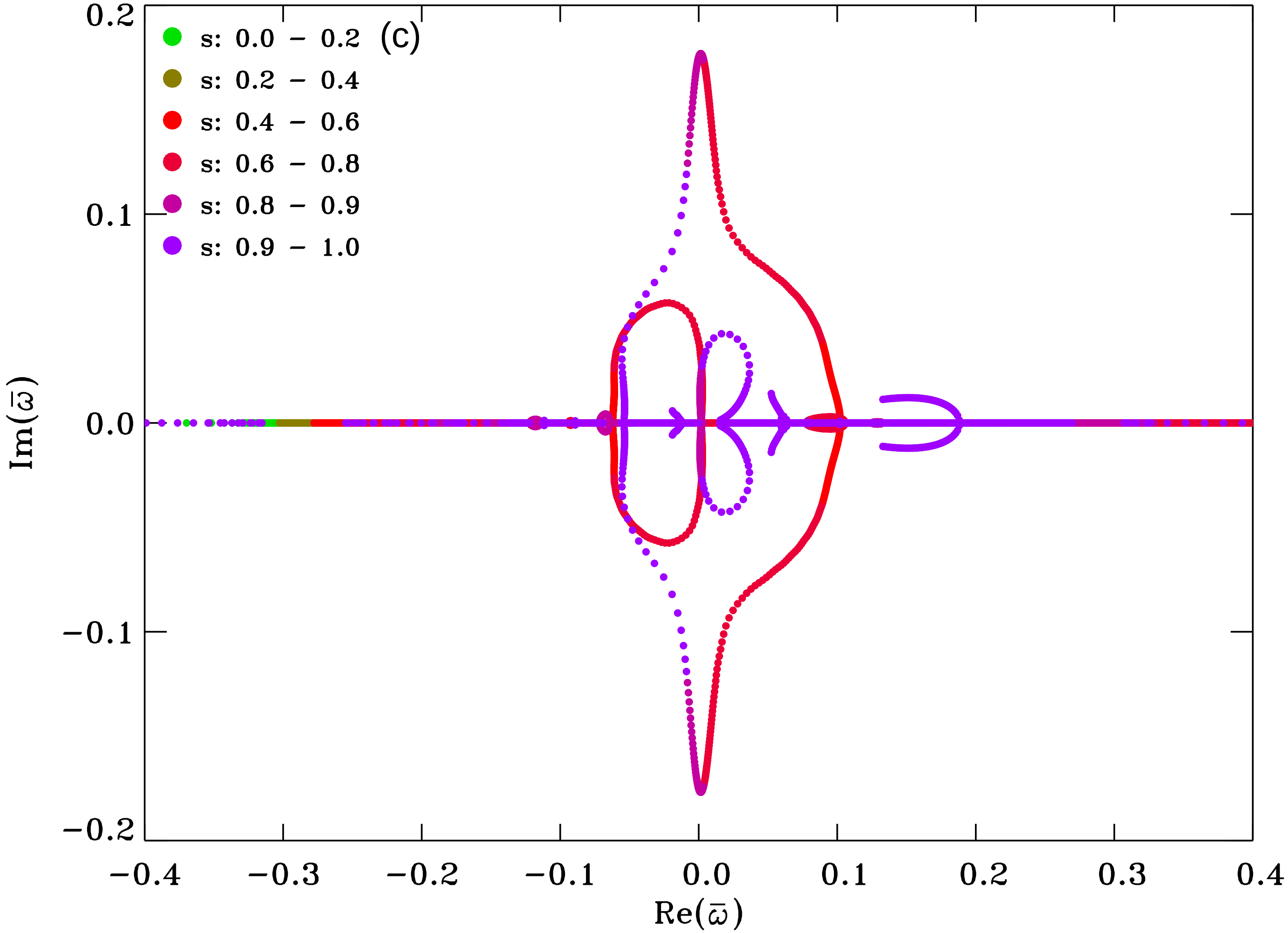
Complex eigenvalues (for $n = 1$), parameterized with flux label $s \equiv \psi^{1/2}$: *Overstable modes rotating clockwise* (Re $\bar{\omega} > 0$), or *anti-clockwise* (Re $\bar{\omega} < 0$).



PHOENIX code [Goedbloed et al., PoP **11**, 28 (2004)]

Accretion disk ($\Gamma = 2$)

Complex eigenvalues ($n = 1$), parameterized with flux $s \equiv \psi^{1/2}$: *Locked modes* ($Re\,\bar{\omega} = 0$) *with huge exponential growth rate!*



PHOENIX code [Goedbloed et al., PoP **11**, 28 (2004)]

Summary on transonic instabilities

- **Instabilities of coupled Alfvén-slow continuous spectra of transonic equilibria become explosive for large central mass.**

- They may cause strong turbulence and anomalous dissipation **facilitating both accretion & ejection of jets** from accretion disks about compact objects.

- They will operate in **any astrophysical system with flow speeds that surpass the slow critical speed.**

- Transonic flow problems demonstrate that the two completely separate activities of **MHD spectroscopy and nonlinear dynamics should be much more integrated.**